Global hyperbolicity & factorization in cosmological models.

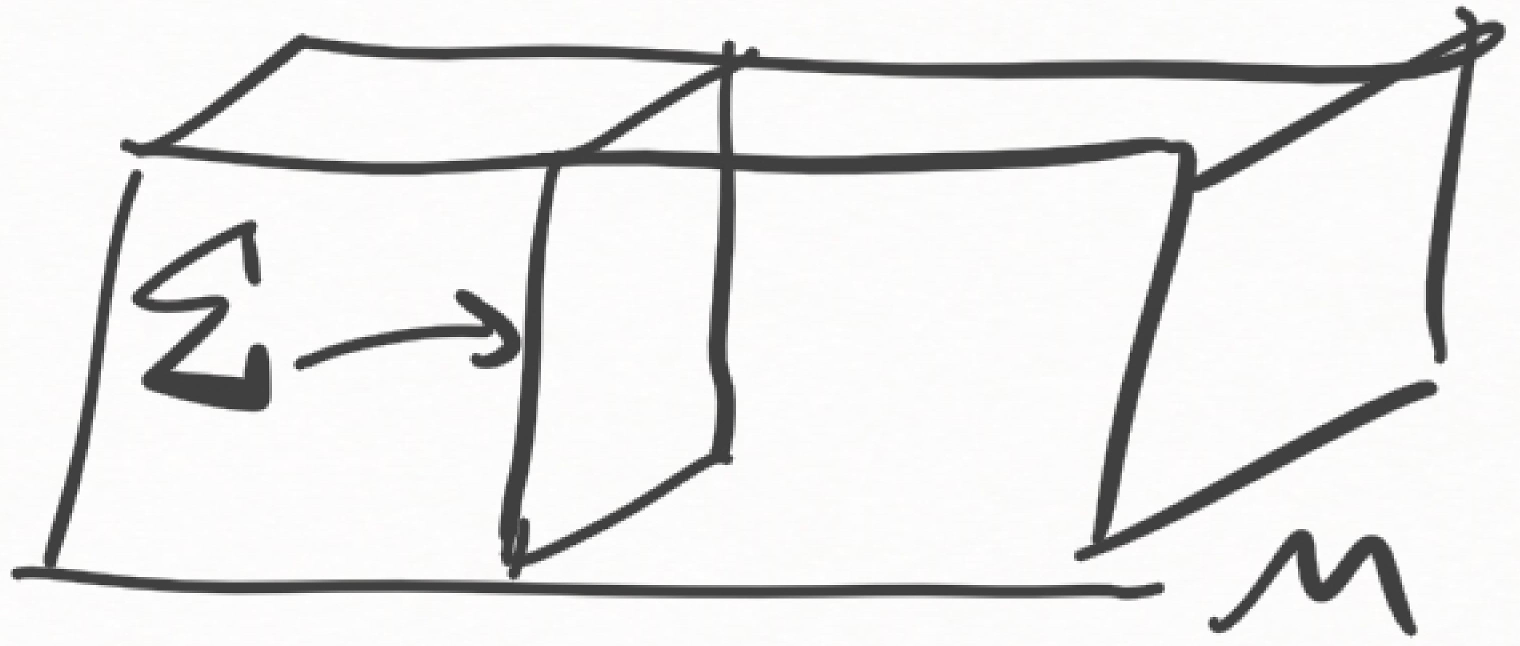
↳ Global hyperbolicity

↳ Homogeneous cosmological spacetimes

↳ Homogeneous cosmological vector bundles

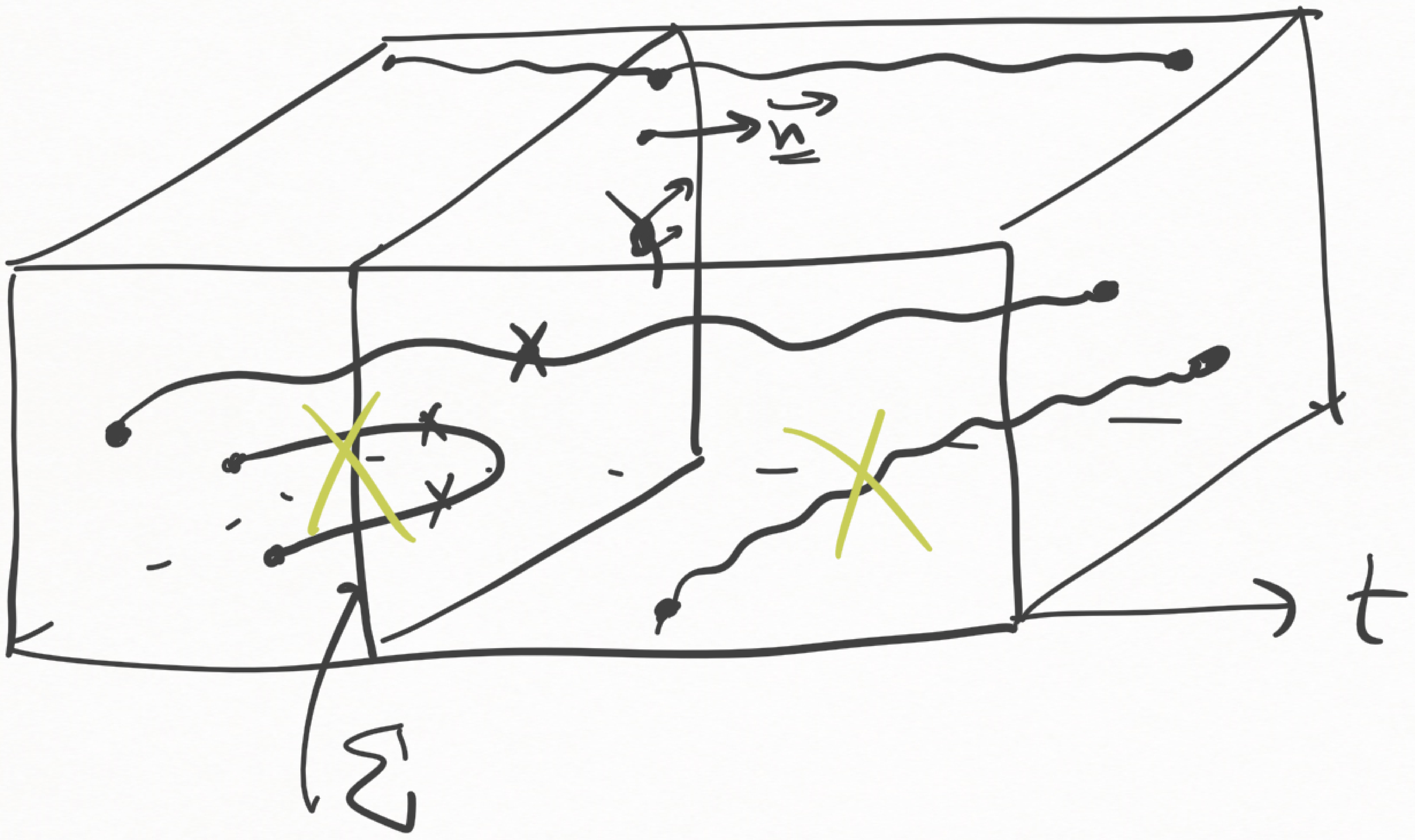
Global hyperbolicity:

- (M, g) connected $d+1$-dim. C^∞ Lorentzian manifold $(+ - - \cdots -)$
- $\partial M = \emptyset$
- Cauchy hypersurface.

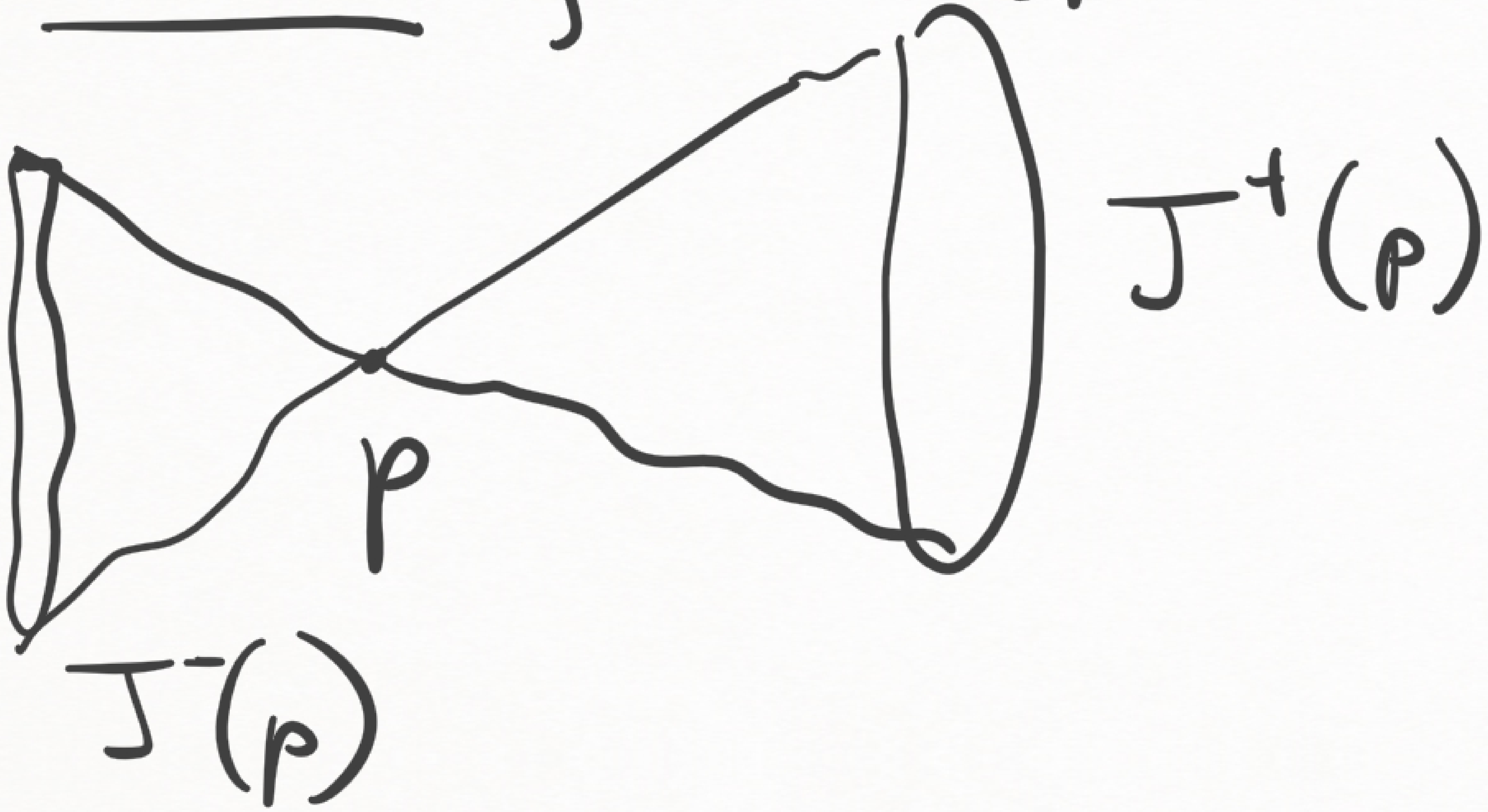


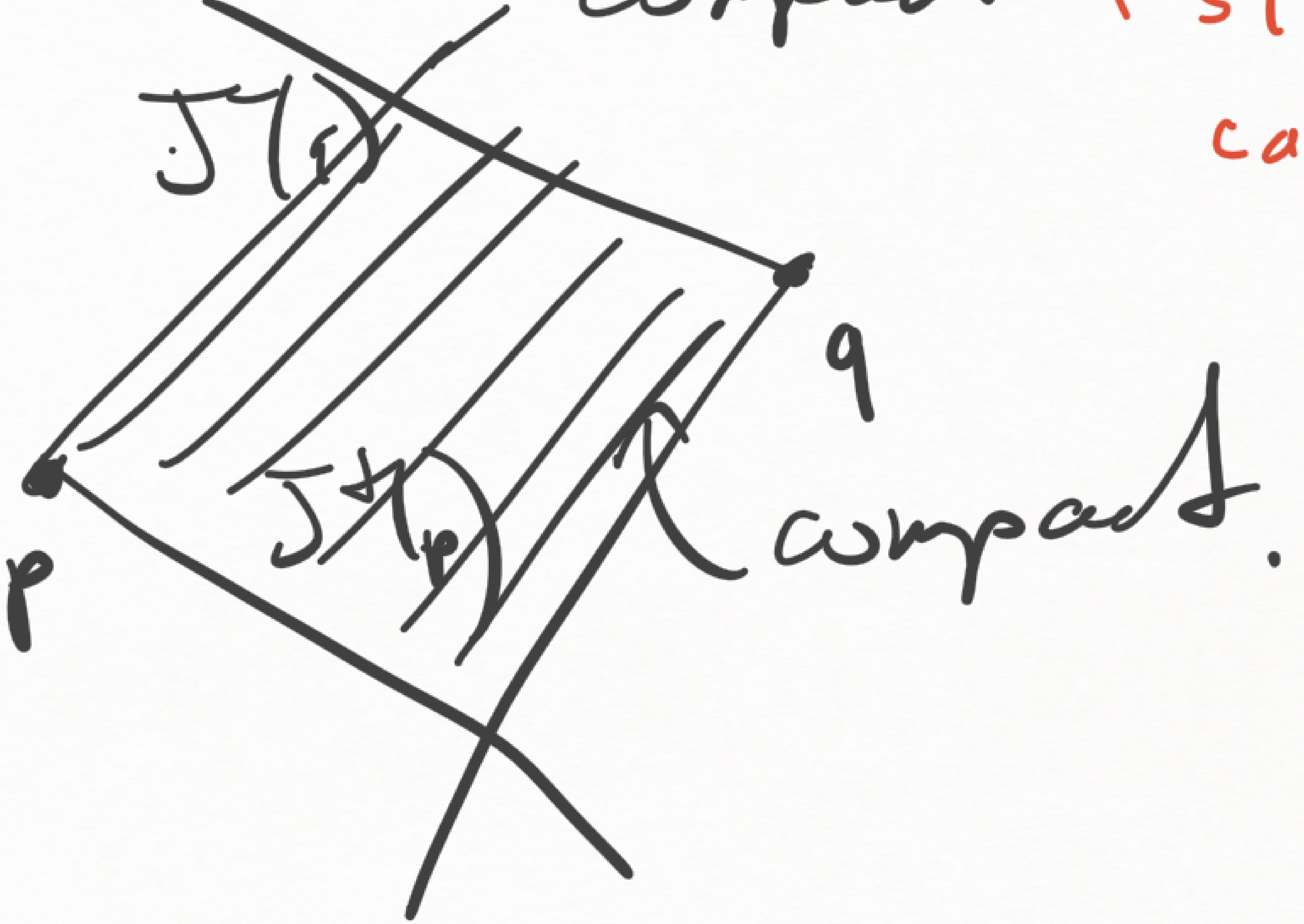
s.d.,

every inextendable causal curve intersects with Σ at exactly 1 point,



Theorem: global hyperbolicity $\Longleftrightarrow$ causal diamonds compact + strongly causal*

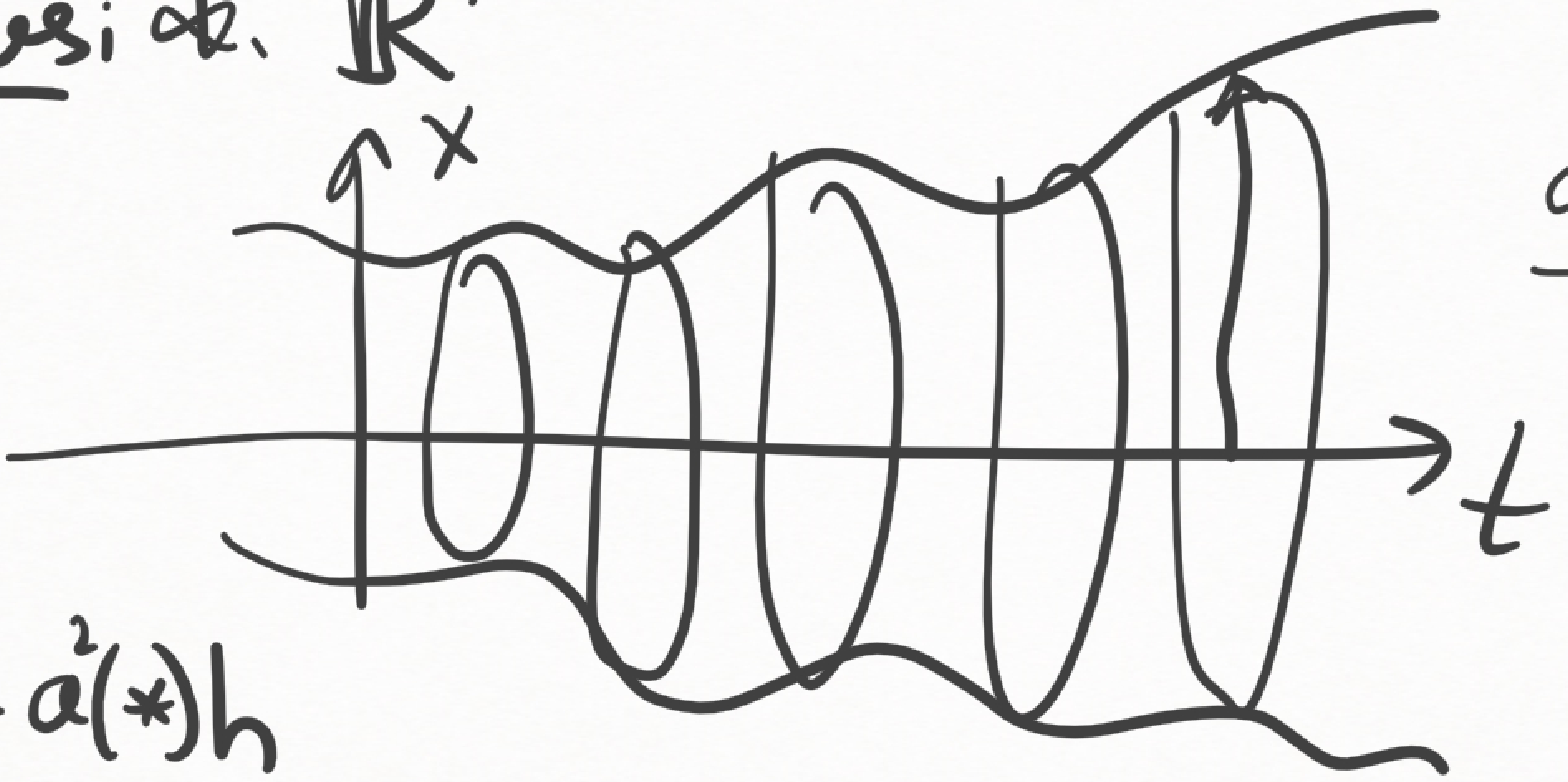




*. See addendum.

Examples. $\mathbb{R}^{1,d}$

*. FRW



a(t)

$g = 1 \oplus -a^2(*)h$

$ds^2 = dt^2 - a^2(t)\,h(dx)$

sphere

*. de Sitter space. $(1-r^2)dt^2 - \frac{dr^2}{1-r^2} - r^2\,dS^2_{n-2}$

$dt^2 - e^{2t}\,dS^2_{n-1}$

*. anti-de Sitter AdS

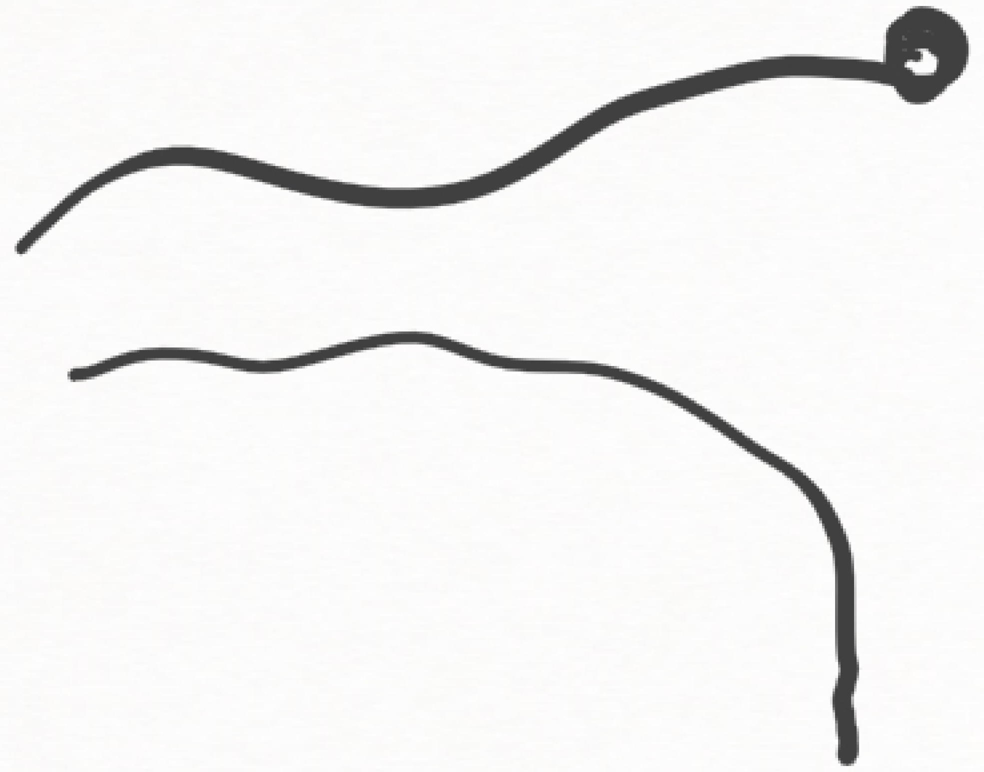
$$(1+r^2)dt^2 - \frac{dr^2}{1+r^2} - r^2 dS_{n-2}^2 \qquad t \in [0, 2\pi)$$

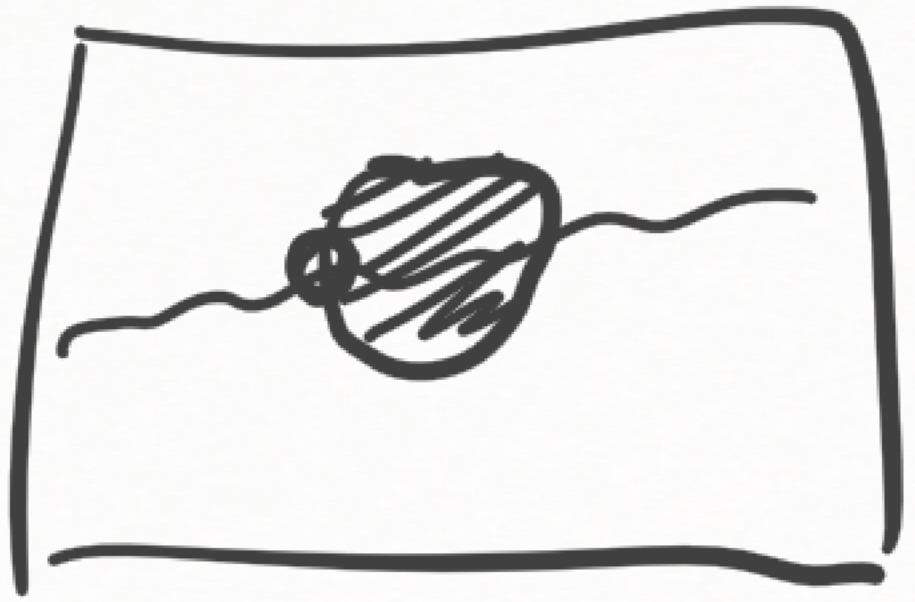
AdS $\longrightarrow t \in \mathbb{R}$.

Reasons to be non-globally hyperbolic

↳ closed causal loops. (∞ intersection)

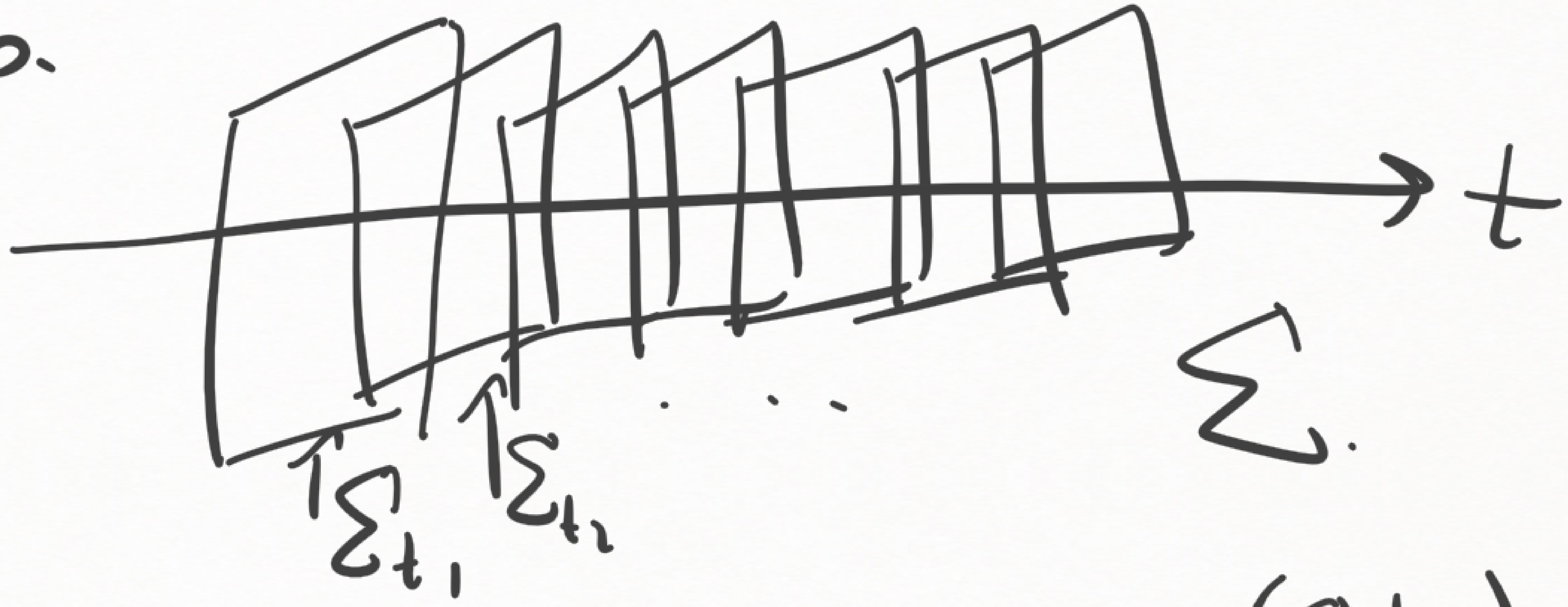
↳ a causal curve has an endpoint (∂)





Consequences of global hyperbolicity

- Geroch (1970) (M, g) glob. hyp. $\Longrightarrow$ $M \overset{\text{homeo}}{\cong} \mathbb{R} \times \Sigma$



- Bernal, Sanchez (2005) (M, g) glob. hyp. $\Longrightarrow$ $(M, g) \simeq (\mathbb{R} \times \Sigma, \beta^2 \oplus -h_*)$

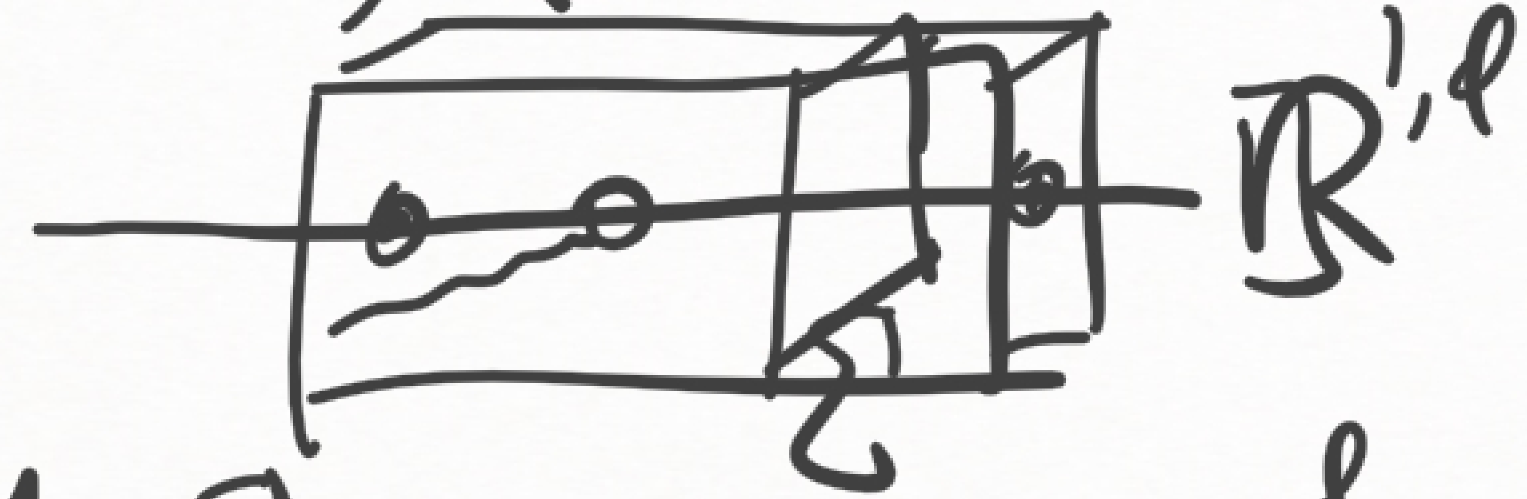
(Σ, h_t) Riemannian

$$ds^2 = \beta^2(t,x)\, dt^2 - h_t(dx)$$

Milestone 1/2: $\underset{\text{glob. hyp.}}{(M,g)} \Longrightarrow (M,g) = (\mathbb{R}\times\Sigma, \beta^2(dt^2 - h_t))$

<u>Question:</u> $(\mathbb{R}\times\Sigma, \beta^2(dt^2 - h_t))$ is this globally hyperbolic?

<u>Answer:</u> <u>No</u>



<u>Question:</u> Which Σ, β, h_t is it glob. hyp.?

- Beem, Ehrlich, Easley (1996): $h_t = f(t)\cdot h$ then (I)

$$\underset{\text{glob. hyp.}}{(\mathbb{R}\times\Sigma, \beta^2(dt^2 - h_t))} \Longleftrightarrow (\Sigma, h) \text{ complete.}$$

• Choquet-Bruhat, Cotsakis (2002)

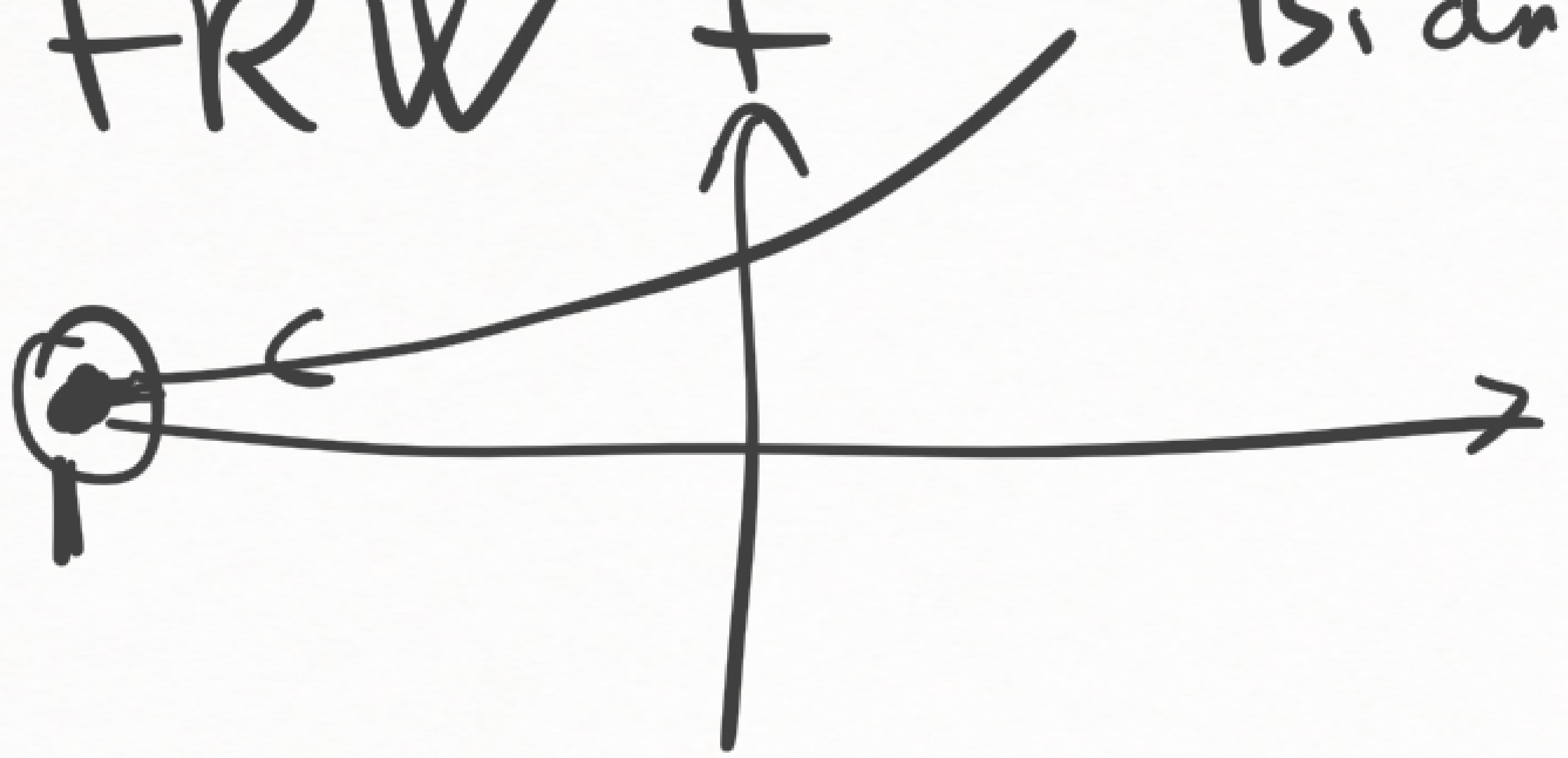
$(\mathbb{R}\times\Sigma, \beta^2\Theta - h_t)$

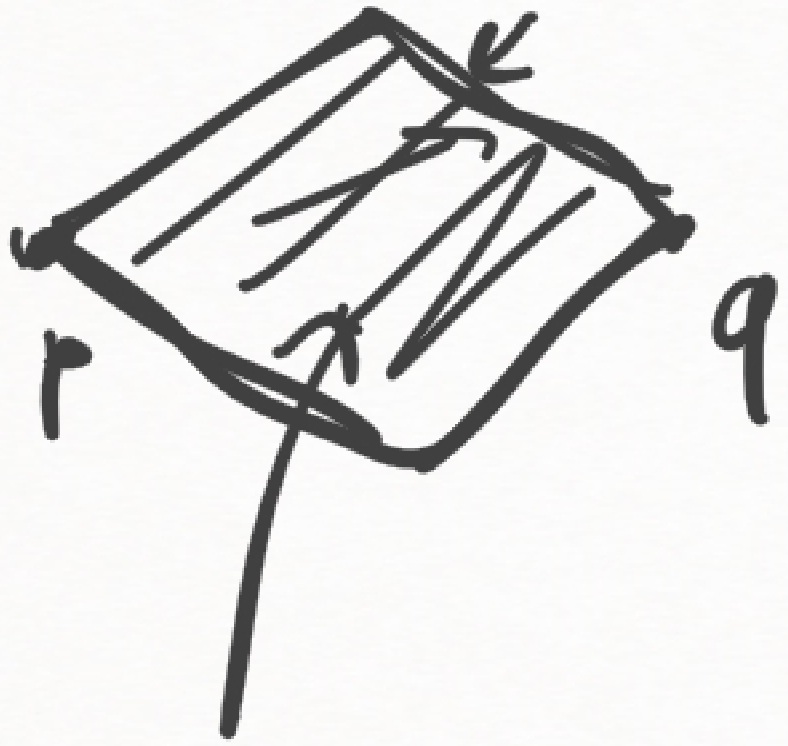
$m \leq \beta(t,x) \leq M \;\wedge\; (\Sigma_0, h_0)$ complete $\wedge \;\inf_{X\in TM} \dfrac{h_t(X,X)}{h_0(X,X)} \geq A$ (A)

$\Longrightarrow$ glob. hyp.

(I) FRW   Bianchi models -

(II)



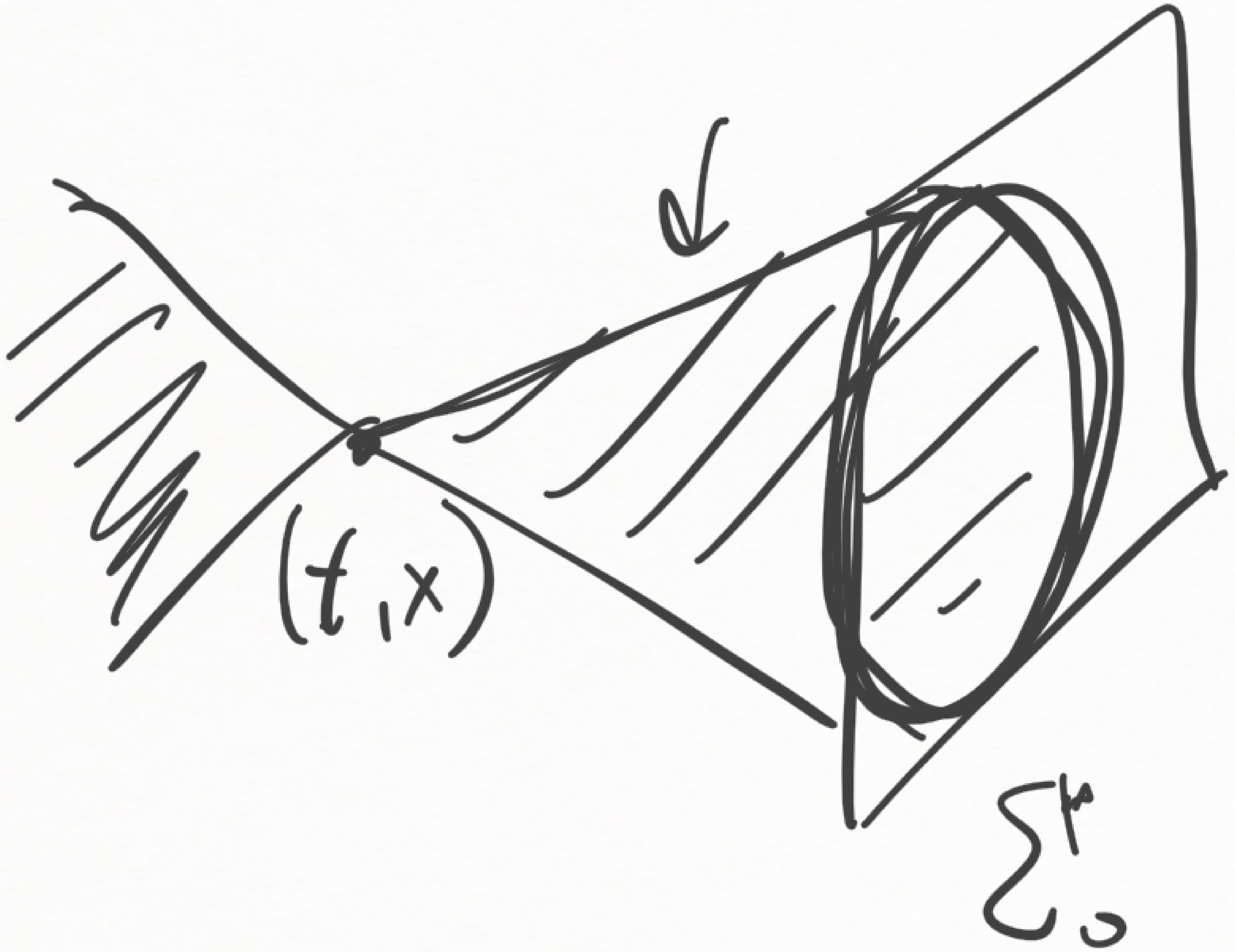


<u>Our approach:</u> Let h_∞ be C^0 Riemannian metric on Σ, (Σ, h_∞) complete. $D : \mathbb{R} \times \Sigma \to \mathbb{R}_+$,

$$D(t,x) \triangleq \frac{\beta^2(t,x)}{\displaystyle\min_{\substack{X \in T_x\Sigma \\ h_\infty[x](X,X)=1}} h_t[x](X,X)} .$$

<u>Proposition:</u> Σ_0 is a Cauchy surface $\iff$

$$\sup_{J^+(t,x)\cap(\mathbb{R}_-\times\Sigma')} D < \infty, \quad \sup_{J^-(t,x)\cap(\mathbb{R}_+\times\Sigma')} D < \infty, \quad \forall (t,x)\in\mathbb{R}\times\Sigma. \qquad (*)$$

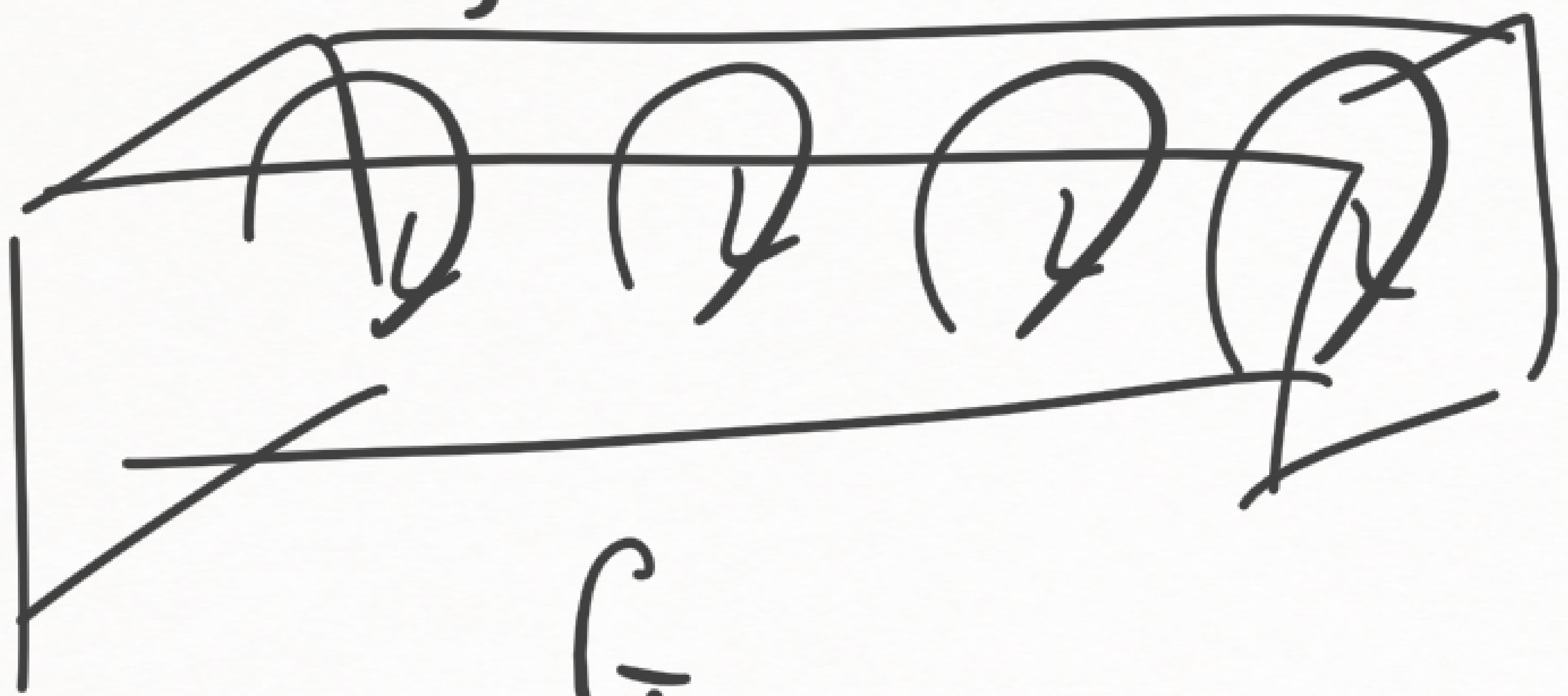


Milestone 2/2: $(M,g) \Longleftrightarrow (M,g) \simeq (\mathbb{R} \times \Sigma^1, \beta^2 (t) - h_x)$

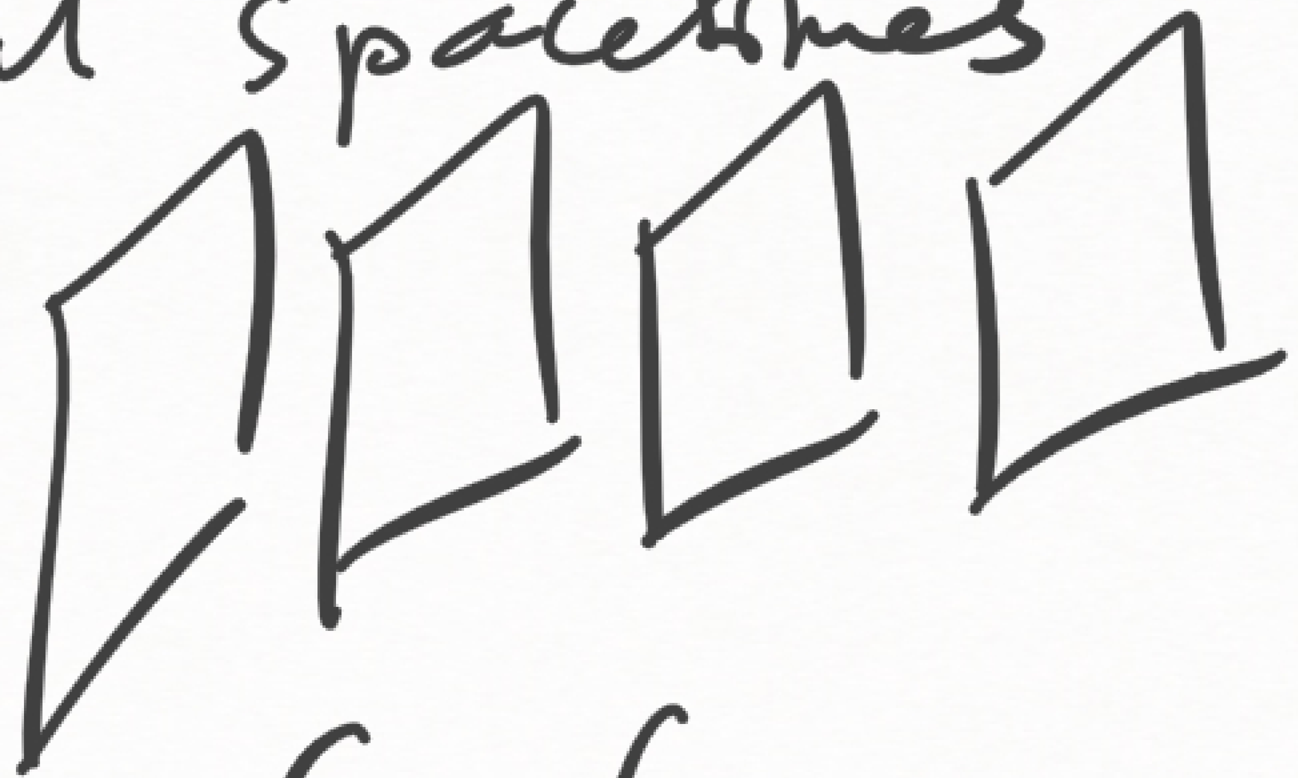
glob. hyp with the conditions $(*)$

&. Covers Bianchi models.

Homogeneous cosmological spacetimes



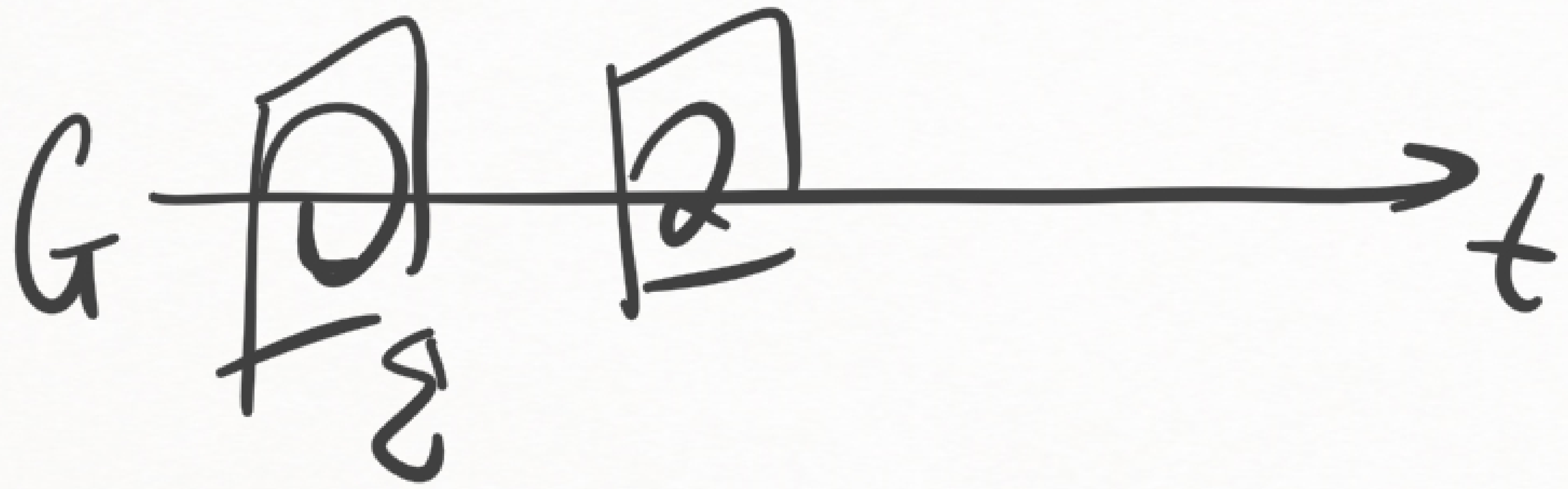
G



Gx Gy Gz ...

<u>Assumptions:</u> G Lie group, (M, g)

$$(G, M, g) \simeq (\mathbb{R} \times (G, \Sigma'), \mathbb{1} \oplus -h_*)$$



Also, Σ'_t Cauchy surface.

- Connected $d+1$-dim C^∞ Lorentzian manifold (M, g)
  $\mathrm{Iso}(M, g)$ isometry group. $G \subseteq \mathrm{Iso}(M, g)$ Lie subgroup.

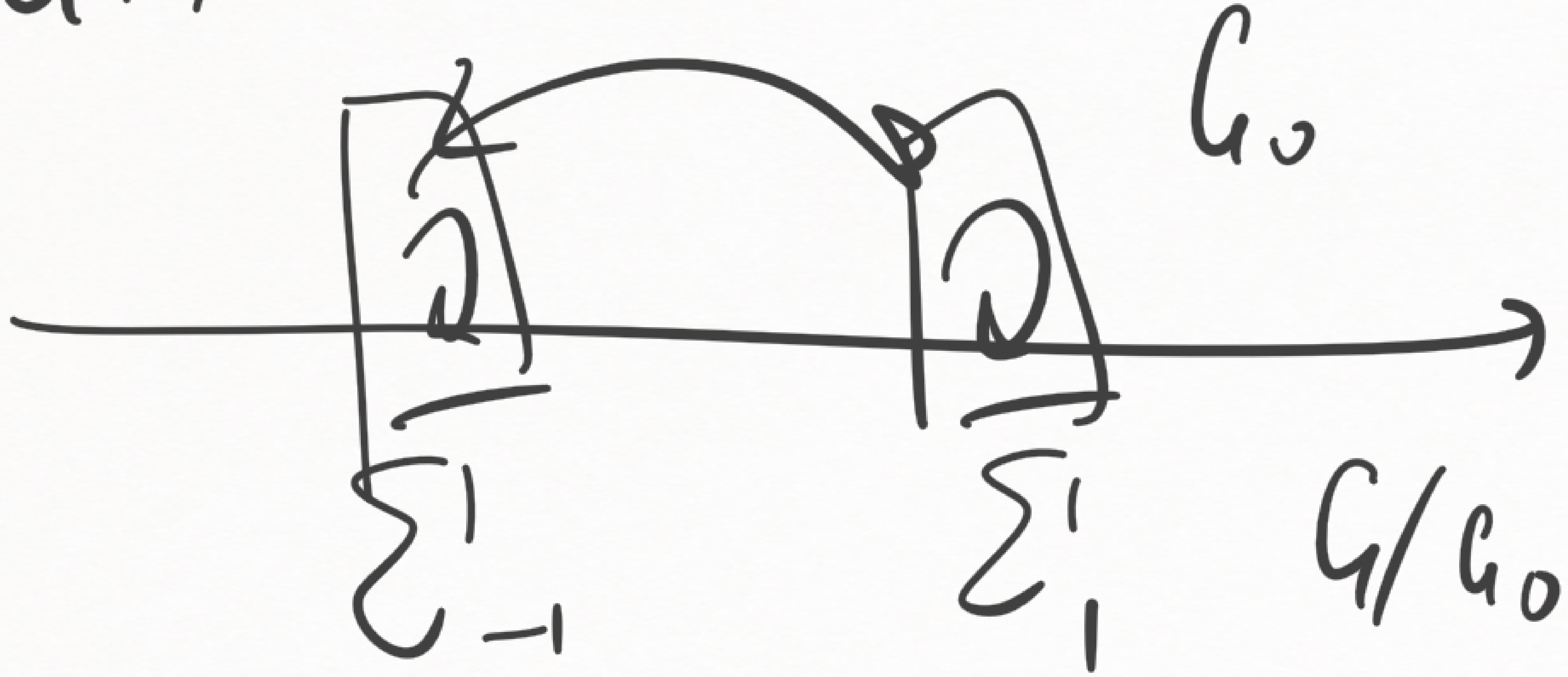
1. G acts on M properly.

2. Killing vector fields of $L(G)$ are spacelike, and the maximal dimension of the distribution they generate is 2.

3. All G-orbits connected.

$$G \times M \ni (g, x) \xmapsto{\phi} (gx, x) \in M \times M$$

proper map, $\phi^{-1}(K) \subset\subset G \times M$ $\forall K \subset\subset M \times M$.



Proposition: All desired properties are met.

Milestone 1/2: $(G, M, g) \Longrightarrow (\mathbb{R} \times (G, \Sigma), 1 \oplus -h_x)$
assumptions

Question: $(\mathbb{R} \times (G, \Sigma), 1 \oplus -h_x) \Longrightarrow$ assumptions?

Proposition: $\forall\, \Sigma = G/H,\ H \leq G$ compact.
Then $(\mathbb{R} \times (G, \Sigma), 1 \oplus -h_x) \Longrightarrow$ assumptions.

Milestone 2/2:

Homogeneous cosmological vector bundles:

$\mathcal{T} \longrightarrow \mathbb{R}\times\Sigma$. $\quad S \doteq \mathcal{T}|_{\Sigma_0}$.

Question: $\mathcal{T} \simeq \mathbb{R}\times S$?

Answer: Topological K-theory.

Question: G acts C^∞ on $\mathcal{T}$, covers G's action on $\mathbb{R}\times\Sigma$, G acts on Σ.* $(G,\mathcal{T}) \simeq (G, \mathbb{R}\times S)$?

Answer: Yes.

*. See addendum

Addendum:

1. On p. 3, the Theorem should read

"globally hyperbolic $\iff$ strongly causal $\wedge$ causal diamonds compact".

Causal diamonds can be defined in non-strongly causal spacetimes, but are not as useful.

Thanks to Jarah Erslin for raising this question.

2. On p. 15, G acts on Σ with compact stability subgroups.