Schrodinger Wave Functionals

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The Idea

Quantization: A map from a classical theory to a quantum theory

In general: It neither exists nor is unique

$\mathcal{C}$ = classical configuration space
$\Rightarrow \{\Psi : \mathcal{C} \to \mathbb{C}\}$ contains the quantum configuration space

Intuitively: $\exists$ a natural embedding $i : \mathcal{C} \hookrightarrow \{\Psi : \mathcal{C} \to \mathbb{C}\}$, taking each $c \in \mathcal{C}$ to a δ-function with support on c

Two tools for studying quantum theories:

1) Semiclassical Expansion: probes a tubular nbd of embedding
$\Rightarrow$ no new topology with respect to the classical theory

2) Instantons: probe a homotopy between two embeddings
$\Rightarrow$ explore the richer and largely unknown topology of the quantum configuration space

Outline: Different Theories

This talk will describe wave function(al)s in two different settings

1) Quantum Mechanics (QM):

   Here, Schrodinger wave functions are maps $\Psi : \mathcal{N} \to \mathbb{C}$

2) Scalar quantum field theory (scalar QFT):

   Let $\mathcal{C}$ be the space of maps $\{\phi : \mathcal{M} \to \mathcal{N}\}$

   Then a *Schrodinger wave functional* is a map $\Psi : \mathcal{C} \to \mathbb{C}$

   Note: This reduces to QM when $\mathcal{M}$ is a point, as $\mathcal{C} \cong \mathcal{N}$

   Note: $\mathcal{C} =$ configuration space in corresponding classical theory

There are other cases of interest in quantum field theory

ex) Yang-Mills theory: $\mathcal{C}$ is the space of principal G bundles over $\mathcal{M}$

ex) QCD: $\mathcal{C}$ is as in Yang-Mills plus one includes an associated vector bundle with a choice of section

Outline: Objects in Each Theory

In each theory will define the following objects

1) The operators: A $\star$-algebra $\mathcal{A}$ over $\mathbb{C}$

2) A (projective) module V of $\mathcal{A}$

   The elements will be in one-to-one correspondence with Schrodinger wave functionals

3) The Hamiltonian: A distinguished self-adjoint operator $\mathrm{H} \in \mathcal{A}$

4) The states: A subspace $\mathcal{H} \subset V$ spanned by eigenvectors of H with finite eigenvalues

   We would like $\mathcal{H}$ to be closed under the action of $\mathcal{A}$ and its spectrum to be bounded from below

The basic problem:
Find the Schrodinger wave functionals corresponding to $\mathcal{H}$

(1-dimensional) Quantum Mechanics: The Algebra

The starting point will always be an algebra $\mathcal{A}$ of operators over $\mathbb{C}$

In quantum mechanics (QM):

Three (self-adjoint) generators: Φ, Π (usually called x and p) and $\hbar$

Relation: $[\Phi, \Pi] = i\hbar$ (called the canonical commutation relation)

$\hbar$ is in the center and powers of $\hbar$ define a grading on everything called the *semiclassical expansion*

Following the usual convention, we often set $\hbar = 1$ and forget the grading, as it is generally not hard to put it back when needed

1d Quantum Mechanics: Comments

Note: This choice of operators is called the *Schrodinger picture*. There are other pictures in which the operators depend on a real parameter called time. We will always use the Schrodinger picture.

Recall the notation $\mathcal{C} = \{\mathcal{M} \to \mathcal{N}\}$ where $\mathcal{M}$ is a point in QM

Note: This QM called 1d because $\mathcal{N}$ is a 1-manifold, as it will be throught the talk.

When we discuss QFT we will use the same notation for the dimension of $\mathcal{M}$, which is 0 for QM and 1 for the QFT examples.

1d Quantum Mechanics: Wave Functions

Next ingredient: a vector space V where the states will live

For every $\phi \in \mathbb{R}$ define an object $|\phi\rangle$

The vector space V is formed of formal linear combinations $|\Psi\rangle$ of these objects with complex coefficients $\Psi(\phi) : \mathbb{R} \to \mathbb{C}$

$$|\Psi\rangle = \int d\phi \Psi(\phi)|\phi\rangle$$

V admits an $\mathcal{A}$-action

$$\Phi|\Psi\rangle = \int d\phi \phi \Psi(\phi)|\phi\rangle, \quad \Pi|\Psi\rangle = -i\hbar \int d\phi \frac{\partial \Psi(\phi)}{\partial \phi}|\phi\rangle$$

$\Psi(\phi) : \mathbb{R} \to \mathbb{C}$ is the Schrodinger wave function

Schrodinger wave functions are in one-to-one correspondence with elements of V, and so also admit an $\mathcal{A}$-action

$$\Phi\Psi(\phi) = \phi\Psi(\phi), \quad \Pi\Psi(\phi) = -i\hbar \frac{\partial \Psi(\phi)}{\partial \phi}$$

1d Quantum Mechanics: Wave Functions to States

The space of wave functions is bigger than the space of states:

States correspond to equivalence classes $\Psi(\phi) \sim e^{i\theta}\Psi(\phi)$

One chooses a lift of the action of $\mathcal{A}$ on wave functions to an action on these equivalence classes (a projective representation)

Various restrictions can be also imposed, depending on the context:

The space of wave functions has a norm

$$\langle \Psi_1 | \Psi_2 \rangle = \int d\phi \Psi_2^*(\phi)\Psi_1(\phi)$$

Often one restricts to states with norm 1.

If $\mathcal{N} \cong S^1$ with circumference R, then $\phi \sim \phi + R$ so one imposes

$$\Psi(\phi + R) = \Psi(\phi)$$

Note that only a subset of $\mathcal{A}$ preserves these restrictions

1d Quantum Mechanics: Hamiltonian

The next step in the construction of quantum mechanics is a choice of Hamiltonian $\mathrm{H} \in \mathcal{A}$

As our study is motivated by Yang-Mills theory, we will imposed some simplifying assumptions on H which are satisfied there

1) H is self-adjoint

2) H preserves any equivalence relations on ϕ such as $\phi \sim \phi + R$
(These equivalences are called *gauge transformations*)

3) The spectrum of H is bounded from below (wlog the lowest eigenvalue is zero, since we can add a constant)

If there are inequivalent choices of irreducible representation of the canonical commutation relations (this only happens in QFT), choose one that satisfies (3)

Example 1: Free Theory

Consider the example

$$\mathrm{H} = \frac{\Pi^2}{2}$$

This acts on a wave function $\Psi(\phi)$ as

$$\mathrm{H}\Psi(\phi) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial\phi^2}\Psi(\phi)$$

The eigenstates are

$$\Psi(\phi) = e^{ip\phi}, \quad p \in \mathbb{R}.$$

If $\mathcal{N} \cong \mathbb{R}$ they are not normalizable, if $\mathcal{N} \cong S^1$ then $p \in (2\pi/R)\mathbb{Z}$ and they are normalizable.

In either case the energy is

$$E = \frac{\hbar^2 p^2}{2}$$

Note that our basis elements $|\phi\rangle$ are not in the span of the finite-energy states

Example 2: Quantum Harmonic Oscillator

$$\mathrm{H} = \frac{\Pi^2 + m^2\Phi^2}{2}$$

Construct the operators

$$a = \sqrt{\frac{m}{2}}\Phi + \frac{i}{\sqrt{2m}}\Pi, \qquad a^\dagger = \sqrt{\frac{m}{2}}\Phi - \frac{i}{\sqrt{2m}}\Pi$$

Note that

$$\mathrm{H} = m\left(a^\dagger a + \frac{\hbar}{2}\right), \quad [a, a^\dagger] = \hbar, \quad [H, a^\dagger] = m\hbar a^\dagger$$

Define the ground state (minimum energy state) $|0\rangle$ by

$$a|0\rangle = 0.$$

Then for all nonnegative integers n, the eigenstates are $a^{\dagger n}|0\rangle$

$$Ha^{\dagger n}|0\rangle = m\hbar\left(n + \frac{1}{2}\right)a^{\dagger n}|0\rangle$$

Quantum Harmonic Oscillator Wave Functions

The ground state

$$|0\rangle = \int d\phi \Psi_0(\phi)|\phi\rangle$$

is annihilated by

$$a = \sqrt{\frac{m}{2}}\phi + \frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial\phi}$$

and so the ground state wave function is

$$\Psi_0(\phi) = e^{-m\phi^2/(2\hbar)}$$

The *n*th excited state is

$$\Psi_n(\phi) = a^{\dagger n}\Psi_0(\phi), \quad a^\dagger = \sqrt{\frac{m}{2}}\phi - \frac{\hbar}{\sqrt{2m}}\frac{\partial}{\partial\phi}$$

Dimensional Analysis

We set the speed of light $c = 1$

This leaves two independent kinds of dimensions: length ℓ and action a

These provide two gradings preserved in every expression

Starting from (We use square brackets both for dimensions and commutators)

$$[m] = \ell^{-1}, \quad [\mathrm{H}] = \ell^{-1} a, \quad \mathrm{H} = \frac{\Pi^2 + m^2 \Phi^2}{2}$$

one finds

$$[\Pi] = \ell^{-1/2} a^{1/2}, \quad [\Phi] = \ell^{1/2} a^{1/2}$$

which is consistent with $[\phi, \pi] = i\hbar$ and $[\hbar] = a$

Example 3: An Interaction: Dimensional Analysis

Consider

$$\mathrm{H} = \frac{\Pi^2 + m^2\Phi^2}{2} + \lambda\Phi^n, \;\; n > 2$$

Proceeding as before

$$[\lambda] = a^{1-n/2}\ell^{-1-n/2}, \;\; [\lambda\hbar^{n/2-1}m^{-n/2-1}] = a^0\ell^0$$

The *semiclassical* expansion = a grading in powers of $\hbar$
A *perturbative* expansion = a grading in powers of λ

There is only one dimensionless quantity $\lambda\hbar^{n/2-1}m^{-n/2-1}$
$\Rightarrow$ once any quantity is graded in terms of ℓ and a, then at each level it is a function $f(\lambda\hbar^{n/2-1}m^{-n/2-1})$

An expansion of f in terms of λ or $\hbar$ is therefore identical
$\Rightarrow$ the semiclassical and perturbative expansions are equivalent

wlog we may set $\hbar = 1$ and consider the expansion in λ

To solve this, expand everything in terms of the λ grading

$$\mathrm{H} = \mathrm{H}_0 + \lambda \mathrm{H}_1, \quad H_0 = \frac{\Pi^2 + m^2\Phi^2}{2}, \quad H_1 = \Phi^n, \; E = \sum_{i=0}^{n} \lambda^i E_i$$

$$|0\rangle = \sum_{i=0}^{n} \lambda^i |0\rangle_i, \; |0\rangle_i = \int \Psi_i(\phi)|\phi\rangle$$

The eigenvalue equation $H|0\rangle = E|0\rangle$ is the recursion relation

$$(H_0 - E_0)|0\rangle_i = -H_1|0\rangle_{i-1} + \sum_{j=1}^{i} E_j|0\rangle_{i-j}$$

The initial condition is just example 2 (solved already)

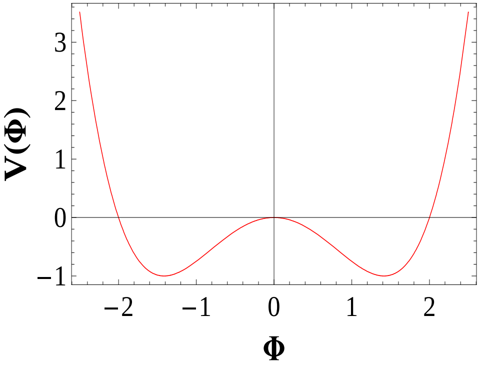
The recursion is trivial because $H_0 - E_0$ is positive-definite/invertible except for its kernel $|0\rangle_0$

To deal with the kernel, impose ${}_i\langle 0|0\rangle_0 = 0$ (This can always be done by shifting E_i)

Example 4: The Double Well: Definition

Lets consider the same Hamiltonian, but changing the sign of the quadratic term (choose $n = 4$ for concreteness)

$$\mathrm{H} = \mathrm{H}_0 + \lambda \mathrm{H}_1, \quad H_0 = \frac{\Pi^2 - m^2\Phi^2}{2}, \quad H_1 = \frac{\Phi^4}{4}$$



Example 4: The Double Well: False Vacuum

Now the initial condition of our perturbation corresponds to the Hamiltonian

$$H_0 = \frac{\Pi^2 - m^2\Phi^2}{2}$$

Its spectrum is not bounded from below $\Rightarrow$ its vacuum (lowest eigenstate) is not defined

The problem is that it we are expanding about $\phi = 0$ which corresponds to a maximum, not a minimum, of the potential

$$V(\Phi) = -\frac{m^2\Phi^2}{2} + \frac{\lambda\Phi^4}{4}$$

This is called a false vacuum

Example 4: The Double Well: Reorganization

$V(\phi)$ has minima at

$$\phi = \pm \frac{m}{\sqrt{\lambda}}$$

Defining the shifted

$$\Phi_\pm = \Phi \mp \frac{m}{\sqrt{\lambda}}$$

One can rewrite write a new decomposition of H

$$\mathrm{H} = \mathrm{H}_0 + \sqrt{\lambda}\mathrm{H}_1 + \lambda H_2, \quad H_0 = \frac{\Pi^2 + 2m^2\Phi_\pm^2}{2} - \frac{m^4}{4\lambda}$$

$$H_1 = \pm m\Phi_\pm^3, \quad H_2 = \frac{\Phi_\pm^4}{4}$$

Now one can write the eigenvalue equation as a recursion as in Example 3, and the initial condition is given by H_0 which is essentially that of Example 2

Example 4: The Double Well: Perturbative Conclusions

We sought out to find a perturbative expansion for the double well

Instead we found two

Each gives the vacuum as a state $|0\rangle_0^{\pm}$ which is localized at $\phi = \pm\frac{m}{\sqrt{\lambda}}$ plus corrections at subleading orders

At each order, the two answers have the same energy as $H(\Phi) = H(-\Phi)$ and $\Phi \to -\Phi$ switches them

So perturbation theory tells us that there are two degenerate vacua

Example 4: Perturbative Conclusions are Wrong

Theorem: All normalizable eigenvectors of $H = \Pi^2/2 + V(\Phi)$ are nondegenerate

Sketch of Proof: Consider two eigenvectors with wave functions $\Psi_i(\phi)$. They solve

$$A_i := -\frac{1}{2}\frac{\partial^2}{\partial\phi^2}\Psi_i(\phi) + (V(\phi) - E)\Psi_i(\phi) = 0$$

and so

$$0 = 2A_1\Psi_2(\phi) - 2A_2\Psi_1(\phi) = \partial_\phi(\Psi_1(\phi)\Psi_2'(\phi) - \Psi_2(\phi)\Psi_1'(\phi))$$

Since they are normalizable, $\Psi_i(\infty) = 0$ and finite eigenvalues imply $\Psi_i'(\infty)$ does not diverge. Thus $\Psi_1\Psi_2{}' - \Psi_2\Psi_1{}' = 0$. Dividing by $\Psi_1\Psi_2$ one finds that $\partial_\phi(ln(\Psi_1) - ln(\Psi_2)) = 0$ so $\Psi_1 \propto \Psi_2$

States are equivalence classes under scalar multiplication
$\Rightarrow$ Ψ_1 and Ψ_2 correspond to the same state

Example 4: What Went Wrong?

Our perturbative calculation gave the wrong number of vacua (2, the answer was 1) at each order

It calculated the vacua as

$$|0\rangle^{\pm} = \sum_{i=0}^{\infty} |0\rangle_i^{\pm}$$

where $|0\rangle_0^{\pm}$ exponentially falls away from $\phi = \pm m/\sqrt{\lambda}$.

$H|0\rangle_0^{\pm}$ exponentially falls away from $\phi = \pm m/\sqrt{\lambda}$.

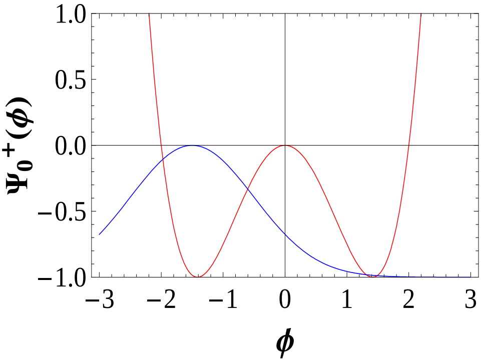
The problem is (after a rotation of $|0\rangle_0^{-}$ to the same phase as $|0\rangle_0^{+}$)

$${}_0^{\mp}\langle 0|H|0\rangle_0^{\pm} = -c, \quad c \in \mathbb{R}^+$$

While $c \sim \exp(-m^3/\lambda)$ is exponentially suppressed in $1/\lambda$, it is nonvanishing

Corrections localized near $\phi = \pm m/\sqrt{\lambda}$ don't cancel this contribution to $H|0\rangle_0^{\pm}$ $\Rightarrow$ can't block diagonalize H

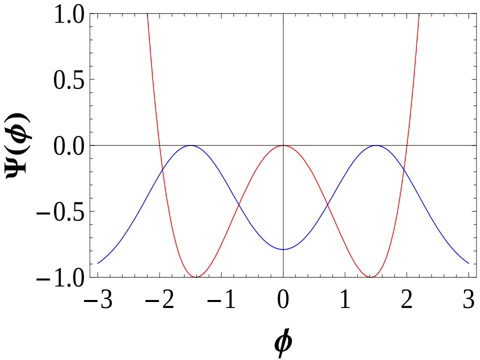
The perturbative approach begins with the solution $\Psi_0^\pm$ of the harmonic oscillator in one well



The nonvanishing matrix element c reflects the fact that $\Psi_0^\pm$ is nonvanishing in the other well

Example 4: Another Viewpoint of $|0\rangle$

In fact, $\Psi_0^{\pm}$ is a terrible approximation for an eigenstate at the other well, as the small potential means that the amplitude of the wave function grows in each well



The even and odd functions, like the even function Ψ shown here, capture this feature

Example 4: How to fix it?

Define

$$|\pm\rangle_0 = |0\rangle_0^+ \pm |0\rangle_0^-$$

then

$$H|\pm\rangle_0 \sim (E \mp c)\,|\pm\rangle_0$$

Therefore the true ground state is $|+\rangle$, while $|-\rangle$ has an energy higher by $O(e^{-m^3/\lambda})$.

This splitting c has an essential singularity at $\lambda = 0$, which led to the failure of our power series approach:

The splitting is not present at any level of the grading.

Said differently: The algebra $\mathcal{A}$ may be graded (by powers of $\hbar$), but V is not

Perturbation theory only probes the graded subspace of V

This subspace does not contain the ground state $|+\rangle$

Example 4: A Systematic Approach?

Given an initial condition for $\Psi(\phi)$ and $\Psi'(\phi)$ in one well, and an energy, one may integrate the equations of motion to find $\Psi(\phi)$ at the other well

Integrating to infinity and checking the boundary conditions determines whether E and the initial conditions were reasonable

This integration from one well to another well requires a path from one well to another well

More generally we will see that we need a homotopy from one minimum to another

This homotopy is called an *instanton*

In quantum mechanics and quantum field theories, such homotopies between minima generate $e^{-1/\hbar}$ effects, which are invisible in our semiclassical expansion (power series in $\hbar$)

The Schrodinger Picture: Operator Algebra

Finally we are ready to study the quantum field theory of a single, real scalar field

We will work in the Schrodinger picture, which we now review

At each point $x \in \mathcal{M}$ ($\mathcal{M}$ is called (physical) space) there are two operators $\Phi(x)$ and $\Pi(x)$ satisfying the canonical commutation relations

$$[\Phi(x), \Pi(y)] = i\hbar\delta(x-y)$$

Our algebra of operators $\mathcal{A}$ is over $\mathbb{C}$ and is generated by $\{\Phi(x), \Pi(x), \hbar\}$

Again $\hbar$ is in the center and will often be set to unity

We will fix $\mathcal{M} \cong \mathbb{R}$, but the generalization of many statements below will be straightforward

Orthonormal Basis of Functions

Consider an orthonormal basis of some space of functions $\{g : \mathcal{M} \to \mathbb{C}\}$, with discrete and continuum elements

$$\int dx g_{k_1}(x) g^*_{k_2}(x) = 2\pi\delta(k_1 - k_2), \qquad \int dx g^*_I(x) g_J(x) = \delta_{IJ}$$

These satisfy completeness relations

$$\sum_I g^*_I(x) g_I(y) + \int \frac{dk}{2\pi} g^*_k(x) g_k(y) = \delta(x-y)$$

For each function, we choose a number ω_k or ω_I and we fix the value of k up to a sign by

$$\omega_k^2 = M^2 + k^2$$

for some M

Let all ω_I be positive except for at most one which is zero, which we call ω_B

Then we can decompose

$$\begin{aligned}
\Phi(x) &= \Phi_B(x)+\Phi_C(x), \qquad \Pi(x)=\Pi_B(x)+\Pi_C(x) \\
\Phi_B(x) &= \Phi_0 g_B(x), \qquad \Phi_C(x)=\int \frac{dk}{2\pi}\frac{1}{\sqrt{2\omega_k}}\left(b_k^\dagger+b_{-k}\right)g_k(x) \\
\Pi_B(x) &= \Pi_0 g_B(x), \qquad \Pi_C(x)=i\int \frac{dk}{2\pi}\sqrt{\frac{\omega_k}{2}}\left(b_k^\dagger-b_{-k}\right)g_k(x).
\end{aligned}$$

We introduced a shorthand where the integral over k includes all discrete I except for B.

The canonical commutation relations imply

$$[\Phi_0,\Pi_0]=i, \quad [b_I,b_J^\dagger]=\delta_{IJ}, \quad [b_{k_1},b_{k_2}^\dagger]=2\pi\delta(k_1-k_2).$$

This is another basis for our operator algebra $\mathcal{A}$

Schrodinger Picture Review: Plane Waves

The simplest example of this decomposition is using plane waves

$$g_k(x) = e^{-ikx}$$

There are no discrete modes

Our decomposition is then just the usual decomposition into the oscillator basis

$$\Phi(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger + a_{-p}\right) e^{-ipx}, \qquad \omega_p = \sqrt{M^2 + p^2}$$
$$\Pi(x) = i \int \frac{dp}{2\pi} \sqrt{\frac{\omega_p}{2}} \left(a_p^\dagger - a_{-p}\right) e^{-ipx}$$

leading to the Heisenberg algebra

$$[a_p, a_q^\dagger] = 2\pi\delta(p - q)$$

Let $\mathcal{C}$ be the space of maps $\{\phi : \mathcal{M} \to \mathbb{R}\}$ and for each element define a vector $|\phi\rangle$.

Let $D\phi$ be a measure on $\mathcal{C}$ (we would like to make sense of this)

Then we can define a vector space by taking the span of all $|\phi\rangle$:

$$|\Psi\rangle = \int_{\mathcal{C}} D\phi \Psi[\phi] |\phi\rangle$$

Here the *Schrodinger wave functional* is $\Psi : \mathcal{C} \to \mathbb{C}$.

Let us define an $\mathcal{A}$-action on $|\Psi\rangle$ or equivalently on $\Psi[\phi]$ by

$$\begin{aligned} \Phi(x) \int_{\mathcal{C}} D\phi \Psi[\phi] |\phi\rangle &= \int_{\mathcal{C}} D\phi \phi(x) \Psi[\phi] |\phi\rangle \\ \Pi(x) \int_{\mathcal{C}} D\phi \Psi[\phi] |\phi\rangle &= -i\hbar \int_{\mathcal{C}} D\phi \frac{\delta \Psi[\phi]}{\delta \phi(x)} |\phi\rangle \end{aligned}$$

Again states will be equivalences classes under a phase shift, so we must lift to a projective representation of $\mathcal{A}$

Examples

In the remainder of the talk, I'll run through the examples above in quantum field theory

We will find the Schrodinger wave functionals in each case

Example 1 turns out to be a bit sick in quantum field theory, so we will begin with Example 2

Example 2: Massive Free Scalar

Consider the Hamiltonian (Integrate over $\mathcal{M} = \mathbb{R}$)

$$\mathrm{H} = \int dx \frac{\Pi^2(x) + (\partial_x \Phi(x))^2 + m^2 \Phi^2(x)}{2}$$

The dimensional analysis is similar

$$[m] = \ell^{-1}, \quad [\mathrm{H}] = \ell^{-1} a, \quad [dx] = \ell$$

leads to

$$[\Pi(x)] = \ell^{-1} a^{1/2}, \quad [\Phi(y)] = a^{1/2}$$

which is compatible with the canonical commutation relations as $[\hbar] = a$

Example 2: Ground State

We can rewrite the Hamiltonian in the $a_p^\dagger$, a_p basis

$$\mathrm{H} = \int \frac{dp}{2\pi} \frac{\omega_p}{2} \left(a_p^\dagger a_p + a_p a_p^\dagger \right) = \int \frac{dp}{2\pi} \omega_p \left(a_p^\dagger a_p + \pi \delta(0) \right)$$

Anything in the kernel of all a_p is our ground state
... but it has energy $\delta(0) \int dp\omega_p/2$

We defined our states as the span of the finite eigenvectors, so this Hamiltonian defines an uninteresting theory with no states

Let's try again

Example 2: Take 2

Let's consider instead the Hamiltonian

$$\mathrm{H} = \int \frac{dp}{2\pi} \omega_p a_p^\dagger a_p$$

We can write this in terms of the fields as follows

$$\mathrm{H} = \int dx \frac{:\Pi^2(x): + :(\partial_x \Phi(x))^2: + m^2 :\Phi^2(x):}{2}$$

where we introduced the normal ordering symbol :: which puts all $a^\dagger$ to the left of a

Again the ground state is the kernel of all a_p

We want the kernel of all operators

$$a_p = \int dx \left(\sqrt{\frac{\omega_p}{2}} \Phi(x) + \frac{i}{\sqrt{2\omega_p}} \Pi(x) \right) e^{-ipx}$$

This looks simpler in terms of the inverse Fourier transformations

$$\tilde{\mathcal{O}}_p = \int dx \mathcal{O}(x) e^{ixp}.$$

Then

$$a_p = \sqrt{\frac{\omega_p}{2}} \tilde{\Phi}_{-p} + \frac{i}{\sqrt{2\omega_p}} \tilde{\Pi}_{-p}$$

which, using $[\tilde{\Phi}_p, \tilde{\Pi}_q] = 2\pi i \delta(p+q)$ acts on a Schrodinger wave functional as

$$a_p = \sqrt{\frac{\omega_p}{2}} \tilde{\phi}_{-p} + \frac{2\pi}{\sqrt{2\omega_p}} \frac{\delta}{\delta \tilde{\phi}_p}$$

Example 2: The Ground State

Summary: For all p, the ground state is annihilated by

$$\omega_p \tilde{\phi}_{-p} + 2\pi \frac{\delta}{\delta \tilde{\phi}_p}$$

The (zero-energy) ground state is therefore (not normalized)

$$\Psi_0[\phi] = \exp\left[-\frac{1}{2}\int \frac{dp}{2\pi}\omega_p \tilde{\phi}_p \tilde{\phi}_{-p}\right]$$

This is called the *vacuum* state

Intuitively, the integral in the exponent means that this is a product of $e^{-\omega\phi^2}$ over p.

This is reasonable: The Hamiltonian is a sum of commuting copies of QM Example 2, one at each p.

$\Rightarrow$ The eigenstates are products of eigenstates of QM Example 2

Example 2: Excited States

$$[H, a_p^\dagger] = \omega_p a_p^\dagger$$

Therefore excited states are obtained by acting monomials in the $a_p^\dagger$ on $\Psi_p[\phi]$

$$a_p^\dagger = \sqrt{\frac{\omega_p}{2}}\tilde{\phi}_{-p} - \frac{2\pi}{\sqrt{2\omega_p}}\frac{\delta}{\delta\tilde{\phi}_p}$$

Example: The state with one particle of momentum q is

$$\Psi_q[\phi] = \tilde{\phi}_{-q} \exp\left[-\frac{1}{2}\int \frac{dp}{2\pi}\omega_p\tilde{\phi}_p\tilde{\phi}_{-p}\right]$$

It has energy ω_q

Example 3: The Φ^4 Theory

Fact: In 1 (spatial)-dimensional scalar quantum field theories, normal ordering eliminates all divergences arising at high energies

Let us now consider the Φ^4 theory

$$\begin{aligned} \mathrm{H} &= \mathrm{H}_0 + \lambda \mathrm{H}_1, \quad \mathrm{H}_1 = \frac{:\Phi^4(x):}{4} \\ \mathrm{H}_0 &= \int dx \frac{:\Pi^2(x): + :(\partial_x \Phi(x))^2: + m^2 :\Phi^2(x):}{2} \end{aligned}$$

Matching levels one sees

$$[\lambda] = \ell^{-2} a^{-1}, \quad \left[\frac{\lambda \hbar}{m^2}\right] = \ell^0 a^0$$

So again a semiclassical expansion in $\hbar$ is equivalent to a perturbative expansion in λ

Example 3: The Ground State

The solution for the ground state is formally identical to Example 3 in quantum mechanics (The text below is copied from above)

The eigenvalue equation is the recursion relation

$$(H_0 - E_0)|0\rangle_i = -H_1|0\rangle_{i-1} + \sum_{j=1}^{i} E_j|0\rangle_{i-j}$$

The initial condition is just example 2 (solved already)

The recursion is trivial because $H_0 - E_0$ is positive-definite/invertible except for its kernel $|0\rangle_0$

To deal with the kernel, impose ${}_i\langle 0|0\rangle_0 = 0$

Example 4: The ϕ^4 Double Well QFT

So far, each QFT has proved to be quite similar to the corresponding QM

The ϕ^4 double-well theory has Hamiltonian

$$H = \int dx \frac{:\Pi^2(x): + :(\partial_x \Phi(x))^2:}{2} + \frac{\lambda}{4} : \left(\Phi^2(x) - v^2\right)^2 :$$

We may follow the same procedure as in QM:

1) Observe that the classical theory has two ground states $\phi(x) = \pm v$
2) Expand the Hamiltonian to quadratic order in Φ and Π about these two ground states, reducing the problem to Example 2
3) Define the corresponding solution from Example 2 as $|0\rangle_0^{\pm}$
4) Define a perturbative expansion, starting with $|0\rangle_0^{\pm}$, using the full Hamiltonian
5) Define the two states $|0\rangle^{\pm}$ to be this perturbative sum

QM: this procedure gave a qualitatively wrong answer for the vacuum

The problem was that

$$ {}^{\mp}_{0}\langle 0|H|0\rangle^{\pm}_{0} = -c, \quad c \sim e^{-S_I/\hbar} $$

This implied that true H eigenstates mix the two sectors, which cannot happen in perturbation theory (polynomials in $\hbar$)

Here S_I is the *instanton action*, which is a measure of the homotopy from the $+$ to the $-$ vacuum

In QFT, this homotopy changes the classical solution $\phi(x)$, or equivalently $\langle 0|\Phi(x)|0\rangle$ at all x

This implies that $S_I = \int dx s_I$ where s_I is independent of x

$\Rightarrow$ $S_I = \infty \Rightarrow c = 0$ $\Rightarrow$ H is block diagonal $\Rightarrow$ two eigenvectors

Example 4: Summary

Summary: The ϕ^4 double-well theory exists in classical physics (0 or 1 spatial dimensions), quantum mechanics (0d) and quantum field theory (1d)

Classical physics: Two solutions, $\phi = \pm v$ in 0d or $\phi(x) = \pm v$ in 1d

Quantum Mechanics: Two solutions in perturbation theory, but due to instantons, only one lowest energy Hamiltonian eigenstate

Quantum Field Theory: Two solutions in perturbation theory, not mixed by instantons as homotopy distance is infinite

$\Rightarrow$ Each classical solution corresponds to a Hamitonian eigenstate (wave functional) in QFT

Example 4: The Classical Kink

We just saw that the two lowest energy classical solution, $\phi(x) = \pm v$, correspond to states in QFT

In 1d, the classical theory has another time-independent solution

The kink:

$$\phi(x) = v \tanh\left(\beta x\right), \quad \beta = \sqrt{\frac{\lambda}{2}} v$$

It interpolates between the two classical vacua $\phi(x) = \pm v$ at $x = \pm\infty$

Does this classical configuration correspond to a Hamiltonian eigenstate in QFT?

Given the Hamiltonian eigenstate $|\Omega\rangle$ corresponding to the classical solution $\phi(x) = 0$ (Such an eigenstate was constructed in Ex. 2) how can we construct a Hamiltonian eigenstate $|K\rangle$ corresponding to a solution to the classical theory $\phi(x) = f(x)$?

Consider the unitary displacement operator

$$\mathcal{D}_f = \exp\left(-i\int dx f(x)\Pi(x)\right)$$

It has the properties (for any functional F):

$$\begin{aligned} F[\Pi(x),\Phi(x)]\mathcal{D}_f &= \mathcal{D}_f F[\Pi(x),\Phi(x)+f(x)] \\ :F[\Pi(x),\Phi(x)]:\mathcal{D}_f &= \mathcal{D}_f :F[\Pi(x),\Phi(x)+f(x)]: \end{aligned}$$

Guess:

$$|K\rangle \stackrel{?}{=} |\mathrm{guess}\rangle = \mathcal{D}_f|\Omega\rangle$$

Example 4: The Kink Hamiltonian

This guess has the suggestive property

$$\langle \text{guess}|\Phi(x)|\text{guess}\rangle = f(x)$$

But it is not a Hamiltonian eigenstate

However it motivates the definition

$$|0\rangle = \mathcal{D}_f^\dagger |K\rangle$$

This has the key property

$$\mathrm{H}'|0\rangle = E|0\rangle \ \Leftrightarrow \mathrm{H}|K\rangle = E|K\rangle$$

where the *Kink Hamiltonian* H' is defined by

$$\mathrm{H}' = \mathcal{D}_f^\dagger \mathrm{H} \mathcal{D}_f$$

Summary: To find an H eigenstate $|K\rangle$ we need only find a H' eigenstate $|0\rangle$ and then act on it with $\mathcal{D}_f$

Example 4: The Pöschl-Teller Hamiltonian

We have reduced the problem of finding eigenstates of the Hamiltonian H to finding eigenstates of the kink Hamiltonian H'

What is the kink Hamiltonian?

$$\mathrm{H}' = Q_0 + \mathrm{H}_{PT} + \mathrm{H}_I, \quad \mathrm{H}_{PT} = \int dx \mathcal{H}_{PT}$$

$$\mathcal{H}_{PT} = \frac{:\pi^2(x):}{2} + \frac{:\partial_x\phi(x)\partial_x\phi(x):}{2} + \left(2\beta^2 - 3\beta^2 \mathrm{sech}^2(\beta x)\right) :\phi^2(x):$$

Q_0 is a (scalar) constant, the classical energy

H_I consists of interactions, which we can add later using perturbation theory

H_{PT} is the Pöschl-Teller Hamiltonian, corresponding to an exactly solvable potential - We want its eigenvectors

This is almost Example 2, except for the x-dependence in the ϕ^2 term

Example 4: Classical Pöschl-Teller Theory

We are looking for eigenvectors of the Pöschl-Teller Hamiltonian

Let us start with the constant-frequency solutions of the classical Pöschl-Teller theory

$$\phi(x,t) = e^{-i\omega t} g(x), \qquad V''[g\Psi(x)]g(x) = \omega^2 g(x) + g''(x).$$

There are continuum solutions $g_k(x)$, $k \in \mathbb{R}$ and two discrete, real solutions $g_S(x)$ ($\omega_S = \sqrt{3}\beta$) and a *zero mode*

$$g_B(x) = \frac{f'(x)}{\sqrt{Q_0}}, \quad \omega_B = 0$$

Fix the parametrization of k, up to a sign, by letting

$$\omega_k = \sqrt{4\beta^2 + k^2}$$

These eigenfunctions are all known in closed form

Example 4: Normal Mode Basis

The functions $g_k(x)$ ($k \in \mathbb{R}$), $g_S(x)$ and $g_B(x)$ are a basis of a space of functions $\mathbb{R} \to \mathbb{C}$

$\Rightarrow$ We can decompose the operators $\Pi(x)$ and $\Phi(x)$ in this basis

Summary: We introduced two new bases of our operator algebra

(The original basis was $\Phi(x)$ and $\Pi(x)$)

$$\begin{aligned} [a_p, a_q^\dagger] &= 2\pi\delta(p-q) \\ [\Phi_0, \Pi_0] &= i, \quad [b_S, b_S^\dagger] = 1, \quad [b_{k_1}, b_{k_2}^\dagger] = 2\pi\delta(k_1 - k_2). \end{aligned}$$

Physical Interpretation:

$a^\dagger$ and a create and destroy plane waves

$b^\dagger$ and b create and destroy kink normal modes

Φ_0 is the position of the center of the kink

Π_0 is the momentum of the center of the kink

Example 4: Quantum Kink State - Conditions

In this basis, H_{PT} is reduced to QM Examples 1 and 2

$$\mathrm{H}_{PT} = \frac{\Pi_0^2}{2} + \omega_S b_S^\dagger b_S + \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k$$

This identifies the ground state $|0\rangle_0$, at leading order in perturbation theory, as the solution to

$$\Pi_0 |0\rangle_0 = b_k |0\rangle_0 = b_S |0\rangle_0 = 0$$

The kernel of Π_0 consists of wave functionals that are independent of ϕ_0.

Example 4: Quantum Kink State - Solution

The condition that $b_k|0\rangle_0 = 0$ is identical to QFT Example 2

The condition $b_S|0\rangle_0 = 0$ is identical to QM Example 2

The solution is then just the product of these examples

$$\Psi_0 = \exp\left[-\frac{1}{2}\phi_S\omega_S\phi_S - \frac{1}{2}\int\frac{dk}{2\pi}\phi_k\omega_k\phi_{-k}\right]$$

$$\phi_S = \int dx\phi(x)g_S(x), \quad \phi_k = \int dx\phi(x)g_{-k}(x)$$

Recall that the kink ground state is not $|0\rangle$ but $|K\rangle = \mathcal{D}_f|0\rangle$, so

$$\Psi_{K_0} = \mathcal{D}_f\Psi_0 = \exp\left[-\frac{1}{2}(\phi_S - f_S)\omega_S(\phi_S - f_S)\right.$$
$$\left.-\frac{1}{2}\int\frac{dk}{2\pi}\left(\phi_k - f_k\right)\omega_k\left(\phi_{-k} - f_{-k}\right)\right]$$

where

$$f_S = \int dxf(x)g_S(x), \quad f_k = \int dxf(x)g_{-k}(x)$$

Example 4: Remarks

We have found the first perturbative approximation to a QFT state corresponding to the kink solution in classical field theory

It looks like a product of Gaussian distributions in an infinite-dimensional space, corresponding to the basis of functions

The Gaussian is centered on the Schrodinger wave functional which is supported on the classical solution, giving zero on all other configurations

In this sense, the support of the wave functions $\Psi_k[\phi]$ is (deformation) retractable to the embedding of the classical solution

$\Rightarrow$ There is no new topology with respect to the classical field theory

Example 4: Higher Orders?

We have recently extended this result to higher orders in perturbation theory

The result is similar to Example 3, one merely multiplies the Gaussians by polynomials

So the support of the wave functional can still be retracted to the classical solution

Example 4: Instantons?

Recall that perturbation theory failed in Example 4 in QM

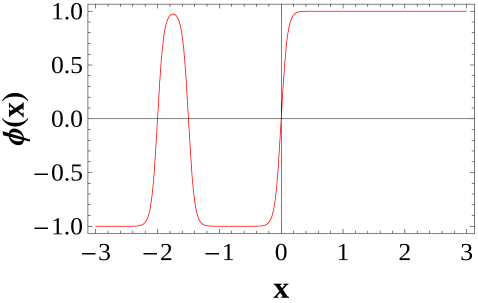
It failed because the wave functions were treated as small perturbations of wave functions localized in one minimum

This approximation fails dramatically once one enters into the other basin of the potential

Here our Gaussians also have tails, which will cross into the other basin

While we have argued that the crossing at all x has a measure zero contribution: *one expects a finite contribution from crossing at $x \in I$ for bounded I*

Example 4: Instanton Cartoon



Here is a picture of such a configuration in $\mathcal{C}$

Since the support of the bump on the left is finite, Ψ_K will not vanish on this configuration

In physics this contribution is an instanton and the ends of the interval are a virtual kink and anti-kink

Computing such corrections to Ψ_K is an open problem

't Hooft-Polyakov Monopole

Consider a classical field theory whose configuration space $\mathcal{C}$ consists of

1) An SU(2) principal bundle over $\mathbb{R}^3$
2) An associated (adjoint rep) vector bundle with a section

Consider a Hamiltonian such that the norm of the section tends to a nonzero constant at infinity, then:

1) Sections are classified by $\pi_2(S^2) = \mathbb{Z}$
2) Each section at infinity is invariant under a $U(1) \subset SU(2)$ whose Chern class (of the $U(1)$ subbundle) is another invariant $\mathbb{Z}$

For Hamiltonians of interest, these two invariants are equal for all finite-energy configurations

If they are both 1, this is called a 't Hooft-Polyakov monopole

Seiberg-Witten SuperQCD

There is another theory, called $\mathcal{N} = 2 \to \mathcal{N} = 1$ superQCD

This theory has a free parameter (called the hypermultiplet bare mass in the physics literature)

For one range of values, the theory has a small coupling constant and ‘t Hooft-Polyakov monopoles

The small coupling constant implies that one can reliably use perturbation theory to calculate the corresponding quantum state

For another range of values:

1) The coupling is large (so instanton corrections are as large as perturbative corrections, meaning that the perturbative expansion is a poor approximation)
2) The section of the associated vector bundle tends asymptotically to zero (So no topological charge and no classical monopole)
3) There is an operator with an expectation value (when acting on the vacuum it gives a state not orthogonal to the vacuum)
4) This operator causes *confinement* (This means that there is no eigenstate corresponding to the $b^\dagger$ operators acting on the vacuum)

In the interval between these two ranges:

1) The monopole operator (at weak coupling $\mathcal{D}_f$) continuously interpolates to the operator that caused confinement
2) It is possible to follow this operator during the interpolation because it is invariant under a certain (super)symmetry

Seiberg-Witten SuperQCD Conclusion

At weak coupling: The quantum configuration space is just a tubular neighborhood of $\mathcal{C}$

It admits a 't Hooft-Polyakov monopole state which deformation retracts to the classical solution

At strong coupling: There is no 't Hooft-Polyakov monopole solution in the classical theory

Yet the quantum configuration space, beyond a tubular neighborhood, still contains a state (and an operator which creates the state from the vacuum) that causes confinement

There is a homotopy between the two configurations

In principle the homotopy can be constructed as it is invariant under a large group of supersymmetries

Summary: Something important, and perhaps very simple, may be lurking in the topology of the quantum configuration space

Summary

Configurations in a classical theory correspond to elts of a set $\mathcal{C}$

Configurations in a quantization of the theory correspond to Schrodinger wave functionals $\Psi : \mathcal{C} \to \mathbb{C}$

Not all maps give states, only the projectivized span of finite Hamiltonian eigenvectors

Perturbation theory probes a tubular neighborhood of an embedding of $\mathcal{C}$ in this larger space

But the poorly understood topology of the larger space plays a key role in the physics, and (at least in superQCD if not ordinary QCD) in the confinement problem

So we should find a suitable definition for this space and understand its topology