Integrable quantum field theories - Inverse scattering, local von Neumann algebras, asymptotic completeness

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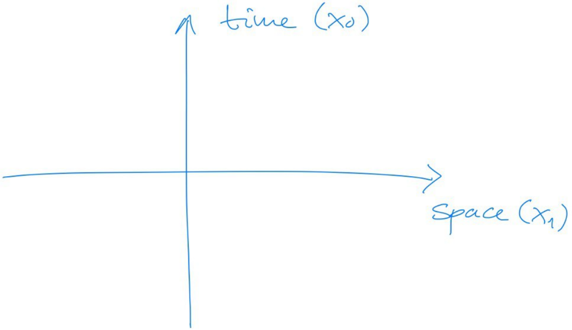
  – Millenium Problem: *Yang–Mills and Mass Gap*
- **This talk:**
  – Overview of non-perturbative construction programme for specific QFT models
  – will focus on QFT on 2d Minkowski spacetime that is "integrable"
  – will use an operator-algebraic setting

QFT on two-dimensional Minkowski space



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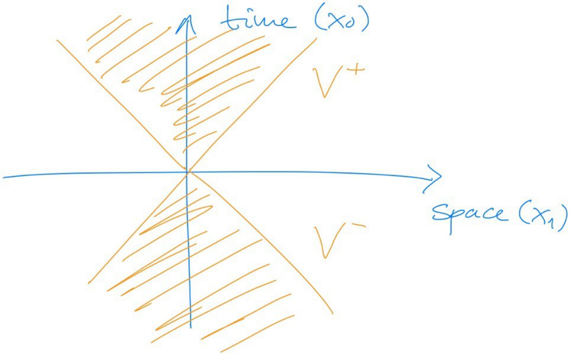

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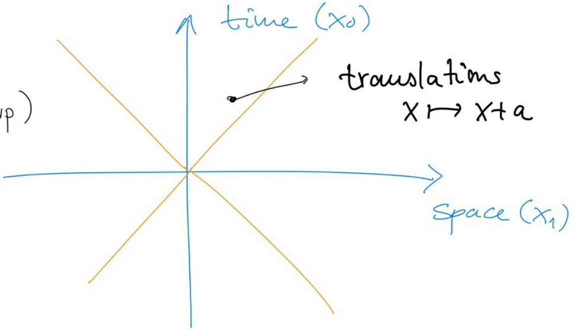
relativistic symmetry: (Poincaré grp) $\mathcal{P}$



translations $x \mapsto x+a$

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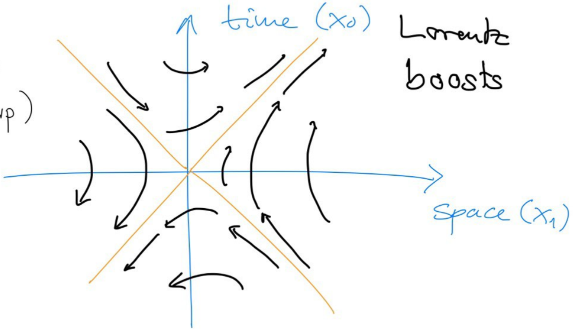

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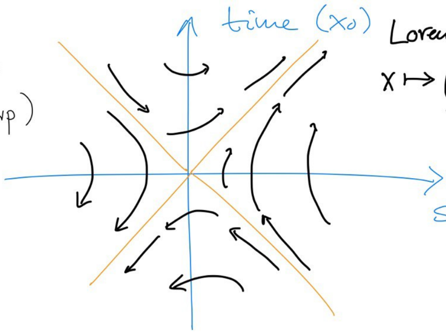

Lorentz boosts: $x \mapsto \underbrace{\begin{pmatrix} \mathrm{ch}\,t & \mathrm{sh}\,t \\ \mathrm{sh}\,t & \mathrm{ch}\,t \end{pmatrix}}_{=:\Lambda_t} x$

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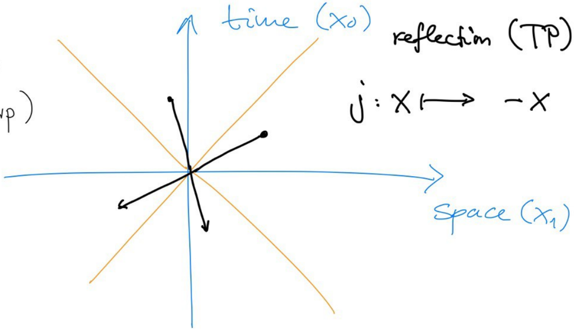
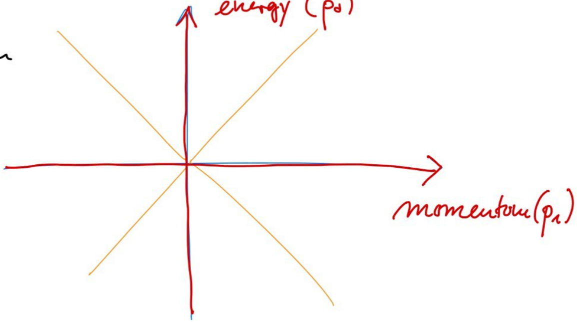


reflection (TP)

$j : x \longmapsto -x$

QFT on two-dimensional Minkowski space

Energy-
momentum
spectrum



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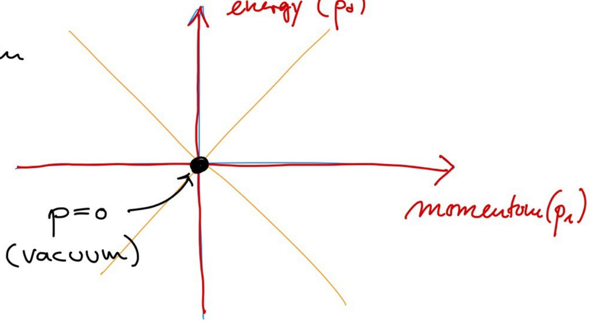

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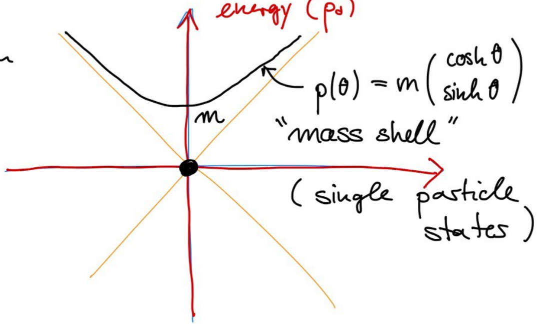

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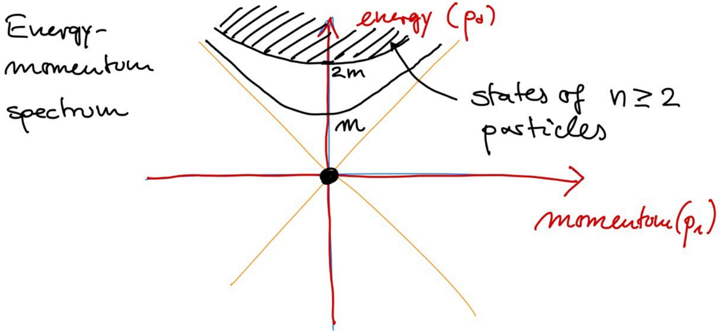

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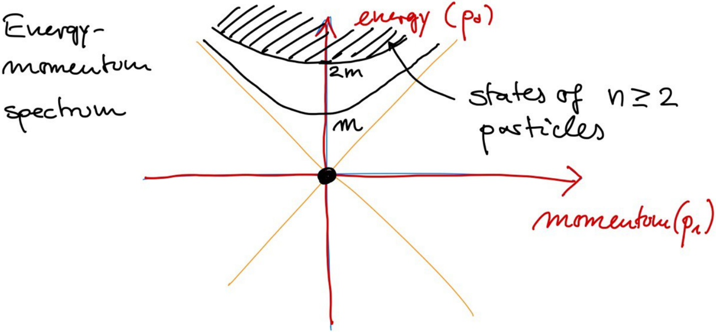

Mathematical input:

- $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$ (single particle Hilbert space),
  $\mathcal{K}$: internal degrees of freedom (could be trivial, $\mathcal{K} = \mathbb{C}$)
- **Unitary positive energy representation** U of Poincaré group $\mathcal{P}$:
  $$(U(x,\lambda)\psi)^\alpha(\theta) = e^{ip(\theta)x}\psi^\alpha(\theta - \lambda), \qquad (U(j)\psi)^\alpha(\theta) = \overline{\psi^{\bar{\alpha}}(\theta)}$$
- n-particle states: suitably symmetrized subspace of $\mathcal{H}_1^{\otimes n}$
- Full Hilbert space $\mathcal{H}$: Fock space $\mathcal{F}(\mathcal{H}_1)$, contains **vacuum vect.** Ω

Integrable scattering

In two dimensions, there exist special *integrable* QFTs with:

- infinitely many conservation laws constraining the dynamics
- No particle production in scattering processes
- $\{$incoming momenta$\}$ = $\{$outgoing momenta$\}$
- n-particle scattering can be described by product of two-particle S-matrices ("factorizing S-matrix")

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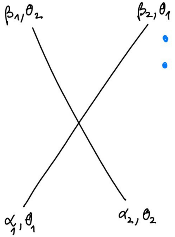

- $\theta_1, \theta_2 \in \mathbb{R}$: rapidities
- $\alpha_i, \beta_i \in \{1, \dots, \dim K\}$: internal d.o.f.

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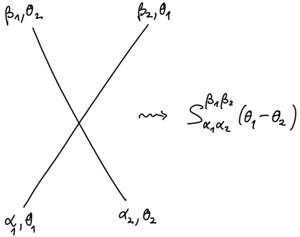

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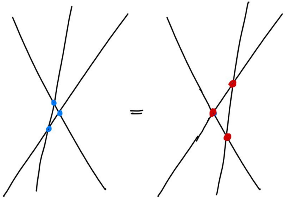

Two-particle S-matrices

Setting: Hilbert space $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}$ with representation U fixed

Definition

A two-particle S-matrix is a continuous function $S : \mathbb{R} \to \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$ s.t.

- $S(\theta)$ is unitary for all $\theta \in \mathbb{R}$
- S solves the factorization (**Yang-Baxter**) equation:

$$(S(\theta) \otimes 1)(1 \otimes S(\theta + \theta'))(S(\theta') \otimes 1) \\ = (1 \otimes S(\theta'))(S(\theta + \theta') \otimes 1)(1 \otimes S(\theta))$$

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$\mathcal{K} = \mathbb{C}^N$, $S(\theta) = \sigma_1(\theta) P + \sigma_2(\theta) 1 + \sigma_3(\theta) F$ $O(N)$-sigma model

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Task: Construct a QFT with a given S as its two-particle S-matrix

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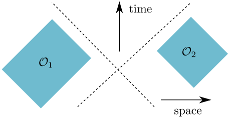

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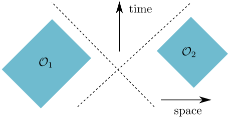

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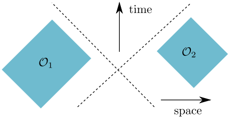

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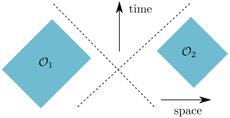

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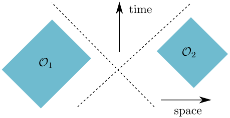

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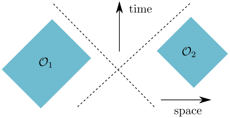

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- **Focus of this talk:** Alternative approach using operator algebras

The inverse scattering construction I: Fock space

Fix one-particle space $\mathcal{H}_1$ and S-matrix S. Let

$$\boldsymbol{S} : \mathcal{H}_1 \otimes \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_1, \qquad (\boldsymbol{S}\Psi_2)(\theta_1, \theta_2) := S(\theta_2 - \theta_1)\Psi_2(\theta_2, \theta_1).$$

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Then $\boldsymbol{S}$ generates representations $\rho_S^{(n)}$ of the symmetric groups $\mathfrak{S}_n$ on $\mathcal{H}_1^{\otimes n}$, fixed on transpositions τ_k via

$$\rho_S^{(n)}(\tau_k) = \boldsymbol{S}_k := 1^{\otimes(k-1)} \otimes \boldsymbol{S} \otimes 1^{\otimes(n-k-1)}.$$

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The S-symmetric Fock space is

$$\mathcal{H}_S := \bigoplus_{n \geq 0} P_S^{(n)} \mathcal{H}_1^{\otimes n}, \qquad P_S^{(n)} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \rho_S^{(n)}(\pi).$$

This is the Bose/Fermi Fock space for $S = \pm F$ (tensor flip).

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On $\mathcal{H}_S$, have creation/annihilation operators $a_S^{\#}(\xi)$, $\psi \in \mathcal{H}_1$:

$$a_S^*(\xi)\Psi_n := \sqrt{n+1} P_S^{(n+1)}(\xi \otimes \Psi_n), \qquad a_S(\xi) = a_S^*(\xi)^*.$$

The inv. scatt. construction II: Wedge-local fields

Consider the field operators

$$\phi_{S,\alpha}(x) := \int_{\mathbb{R}} d\theta \left(e^{ip(\theta)\cdot x} a^*_{S,\alpha}(\theta) + e^{-ip(\theta)\cdot x} a_{S,\bar{\alpha}}(\theta) \right).$$

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- The integrated fields $\phi_S(f) = \sum_\alpha \int d^2x f_\alpha(x) \phi_{S,\alpha}(x)$, $f_\alpha \in \mathscr{S}(\mathbb{R}^2)$, can be given proper meaning as essentially selfadjoint operators on $\mathcal{H}_S$.

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- **Covariance:** We have $U(a,\lambda)\phi_{S,\alpha}(x)U(a,\lambda)^{-1} = \phi_{S,\alpha}(\Lambda_\lambda x + a)$, but **not** $U(j)\phi_{S,,\alpha}(x)U(j)^{-1} = \phi_{S,\bar{\alpha}}(-x)$.

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- **Locality:** We do **not** have $[\phi_{S,\alpha}(x), \phi_{S,\beta}(y)] = 0$ for $x - y$ space-like unless $S = F$.

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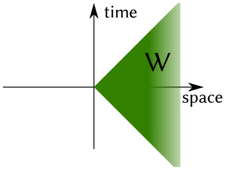
Wedges

Consider the *right wedge*

$$W := \{x \in \mathbb{R}^2 : x_1 > |x_0|\}$$

with causal complement $W' = -W$ (left wedge).



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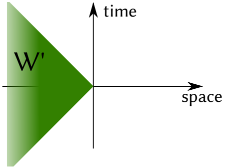

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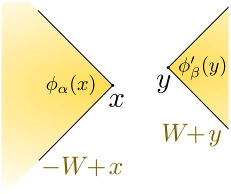

- $\phi_{S,\alpha}(x)$ and $\phi'_{S,\beta}(y)$ commute if the wedge regions $-W+x$ and $W+y$ are space-like separated.
- Interpretation of commutation theorem: $\phi_{S,\alpha}(x)$ is localized in the region $-W+x$, **not** at the point $\{x\}$ (**wedge-local quantum field**)

Operator-algebraic formulation

To proceed from **wedge-local** to **local** observables/fields, it is advantageous to use an operator-algebraic formulation.
Consider the *von Neumann algebras*

$$\mathcal{M}_S(x) := \{\text{bounded measurable functions of } \phi'_S(f),\ \mathrm{supp}(f) \subset W + x\}$$

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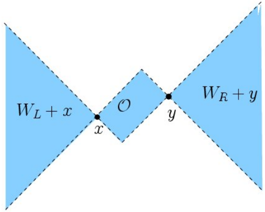
→ setup of algebraic QFT, where one considers nets of von Neumann algebras $\mathbb{R}^2 \supset \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$. **Idea:** $\mathcal{A}(\mathcal{O})$ describes the observables measurable within space-time region $\mathcal{O}$.

Local observables

What is the algebra of all observables localized in a double cone $\mathcal{O}_{x,y}$?



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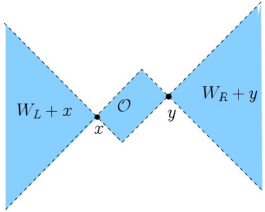

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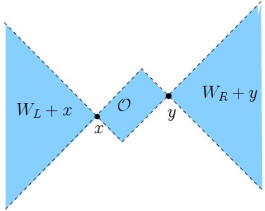

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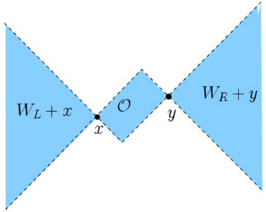

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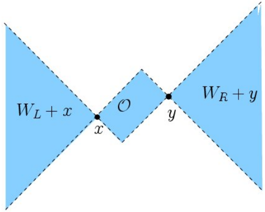

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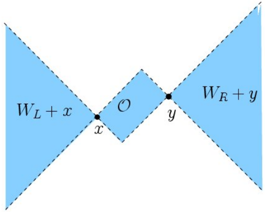

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- Decide non-triviality of $\mathcal{A}_S(\mathcal{O})$ abstractly.

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- If the modular nuclearity condition holds, then our construction produces a well-defined QFT.

Existence Results

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- Theorem applies to various models, e.g. Sinh-Gordon, diagonal S, Ising. Case of $O(N)$-models is still open.

Scattering theory and asymptotic completeness

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and the corresponding model is asymptotically complete. The S-matrix is (for scalar models)

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Proves ideas underlying the Zamolodchikov-Faddeev algebra.

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Extensions/other directions:

- Inclusion of bound states
- Perturbative expansions, contact with form-factor programme
- Contact with non-local models (non-commutative geometry, free probability)
- Characterization of integrability (polarization-free generators)
- Models in higher dimensions