From Orbit to Particle
(and its generalizations)

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based on ongoing work with
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other older works with colleagues (Gomis, Kleinschmidt, Mkrtchyan)

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many classical references

Why particle action ?

- Many different approaches to (spinning) ptcl action
  - bosonic / fermionic spin variables
  - twistor variables
  - orbit method

- All these methods are related

$\Rightarrow$ More systematic and more global view
with some new proposals

Why particle action?

- Understand how a symmetry determines a theory
  - Role of hidden or higher symmetries

- Quantization of ptcl action $\Rightarrow$ "oscillator"
  (Hilbert space)

- How a Hilbert space decomposes differently according to different symmetries
  - Symplectic reduction / Dual pair correspondence

Orbit Method (and "Nonlinear Realization")

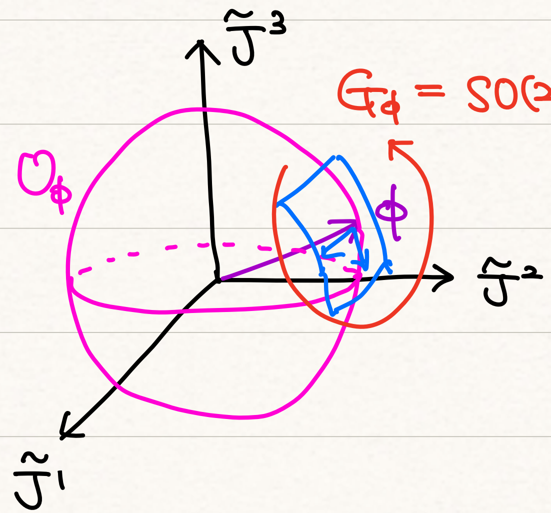
<u>Sketch</u>

$$
\begin{tikzcd}
 & & \text{Lie group } G \arrow[dl, Rightarrow] \arrow[dr, dotted, no head] & & \\
 & \text{Coadjoint orbit} \arrow[r, Rightarrow] \arrow[dl, phantom, sloped, "\simeq"] & & \text{Irrep of } G \arrow[dr, dotted, no head] & \\
\text{Phase space} \arrow[rr, Rightarrow] & & \begin{array}{c}\text{Classical} \\ \text{Particle action}\end{array} \arrow[rr, Rightarrow] & & \begin{array}{c}\text{Quantum} \\ \text{Particle theory}\end{array}
\end{tikzcd}
$$

Coadjoint orbit

ex: $SO(3) = \mathrm{Span}\{J_a\}$ $\quad a = 1, 2, 3$

coadjoint space $\quad so(3)^* = \mathrm{Span}\{\tilde{J}^a\}$ with $\langle \tilde{J}^a, J_b \rangle = \delta^a_b$



$$\phi = \phi_a \tilde{J}^a \in so(3)^*$$

$$\mathcal{O}_\phi = \{ Ad^*_g \phi \mid g \in SO(3) \}$$

$$\langle Ad^*_g \phi, X \rangle = \langle \phi, Ad_{g^{-1}} X \rangle$$

$$\mathcal{O}_\phi \simeq SO(3)/SO(2) \simeq S^2$$

$\mathcal{O}_\phi$ is symplectic : $\quad \omega_\phi(X, Y) = \langle \phi, [X, Y] \rangle$

$$\omega_\phi = \langle \phi, g^{-1}dg \wedge g^{-1}dg \rangle = -d\left(\langle \phi, g^{-1}dg \rangle \right)$$

Coadjoint orbit

in general, $\mathcal{O}_\phi \simeq G / G_\phi$

$g = ab \in G$, $b \in G_\phi$. $[a] = [ab] \in \mathcal{O}_\phi$

$$\langle \phi, g^{-1} dg \rangle = \langle \phi, a^{-1} da \rangle + \langle \phi, b^{-1} db \rangle$$

$$d \langle \phi, a^{-1} da \rangle \neq 0 \quad \text{but} \quad d \langle \phi, b^{-1} db \rangle = 0$$

Mechanical action

$$iS = \int \langle \phi, a^{-1} da \rangle + \underbrace{\int \langle \phi, b^{-1} db \rangle}_{\text{boundary or topological term}}$$

Poincaré group $ISO(1,3)$ with $\phi = m\tilde{P}^0$

$G_\phi = \mathbb{R} \times SO(3) \quad \left(\mathfrak{g}_\phi = \mathrm{span}\{P_0, J_{ab}\}, \quad a,b = 1,2,3 \right)$

- $g = \underbrace{e^{iy^a P_a} e^{iv^a J_{0a}}}_{a} \underbrace{e^{iy^0 P_0} e^{i\omega^{ab} J_{ab}}}_{b}$

$\Rightarrow \langle \phi, g^{-1}dg \rangle = i\left(p_a dy^a + dy^0 \right) \quad \left(p_a = \frac{\mathrm{sh}\, v}{v} v_a \right)$

Only symplectic structure without Hamiltonian

- $g = e^{ix^\mu P_\mu} e^{iv^a J_{0a}} e^{i\omega^{ab} J_{ab}}$

$\Rightarrow \langle \phi, g^{-1}dg \rangle = i\left(p_a dx^a + \sqrt{p^2 + m^2}\, dx^0 \right)$

Coord transf

Symplectic structure with Hamiltonian

Lessons

- Orbit method can dictate both
  - Phase space (symplectic structure)
  - Dynamics (Hamiltonian)

- Dynamics depends on coordinate system

  ⇒ choose a MANIFESTLY COVARIANT coordinate system

  and

  it is convenient to introduce a REDUNDANT one ex: SO(3) again

$\mathcal{O}_\phi \simeq S^2$: Spherical coord of S^2 is not so covariant

Instead, $\tilde{J}^a = \epsilon^{abc} q_b p_c$ is manifestly covariant

However, $\{q_1, q_2, q_3, p_1, p_2, p_3\} \Rightarrow$ 6 d.o.f

+ additional coordinates !

We need to reduce 6 dim phase space

down to 2 dim "physical" phase space

by imposing constraints

Constrained System

$$L\,dt = \underbrace{p_I\, dq^I}_{\text{ambient phase space}} - A^a C_a(q,p) \qquad I = 1, \cdots, N$$

Constraints $\quad C_a(q,p) = J_a(q,p) - \phi_a$

with $\{J_a, J_b\} = f_{ab}{}^c J_c \Rightarrow$ Lie algebra $\mathfrak{g}$

- 1st class constraints : $J_a \in \mathfrak{g}_\phi$ $\rightarrow$ remove $2\dim \mathfrak{g}_\phi$
- 2nd class constraints : $J_a \in \mathfrak{g} \setminus \mathfrak{g}_\phi$ $\rightarrow$ remove $\dim \mathfrak{g} - \dim \mathfrak{g}_\phi$

Altogether remove $\dim \mathfrak{g} + \dim \mathfrak{g}_\phi$

ex: $SO(3)$

$$\left[\begin{array}{ll} \text{1st class constraint :} & C_1 = q\cdot p - \phi \\ \text{2nd class constraints :} & C_2 = q^2, \quad C_3 = p^2 \end{array}\right\rangle \text{Remove } 4 \text{ d.o.f}$$

Gauge symmetry

$$L\,dt = p_I\, dq^I - A^a \left(J_a(q,p) - \phi_a \right)$$

1st class constraints

$$C_\alpha = J_\alpha(q,p) - \phi_\alpha$$

$$\begin{cases} \delta A^\alpha = d\lambda^\alpha + f^\alpha{}_{\beta\gamma} A^\beta \lambda^\gamma \\ \delta q^I = \{ \lambda^\alpha J_\alpha , q^I \} \\ \delta p_I = \{ \lambda^\alpha J_\alpha , p_I \} \end{cases}$$

Finite transf. $\quad A' = -i\, h^{-1} dh + h^{-1} A h$

$$L'\,dt = L\,dt - i \underbrace{\langle \phi , h^{-1} dh \rangle}_{\text{boundary or topological term}}$$

$= 2\pi \times (\text{integer})$

$\Rightarrow$ quantization of ϕ $\quad$ (quantization of spin)

Further gauging

$$L\,dt = p_I D q^I + \langle \phi, g^{-1} D g \rangle$$

$$\begin{cases} p_I D q^I = p_I d q^I - A^a J_a(q, p) \\ g^{-1} D g = g^{-1} d g + g^{-1} A g \end{cases}$$

with $g' = h^{-1} g$ in addition

Physical d.o.f : $2N + \underbrace{\dim \mathcal{O}_\phi - 2 \dim \mathcal{G}}_{-(\dim \mathcal{G} + \dim \mathcal{G}_\phi)}$

There are at least two ways to obtain manifestly covariant phase space from a coadjoint orbit of G

① Cotangent bundle T^*G as ambient phase space

Spinning top model

Physical d.o.f : $2N - (\dim \mathfrak{g} + \dim \mathfrak{g}_\phi) = \dim \mathfrak{g} - \dim \mathfrak{g}_\phi$

$= \dim \mathcal{O}_\phi$

② $\mathbb{R}^{2N}$ as ambient phase space

Twistor & other spinning ptcl models

$\mathbb{R}^{2N}/\mathbb{Z}_2$: minimal orbit of $Sp(2N,\mathbb{R})$

J_a forms a "dual" Lie algebra $\tilde{\mathfrak{g}}$

where $\underbrace{(G, \tilde{G})}_{\text{reductive dual pair}} \subset Sp(2N,\mathbb{R})$ are mutual stabiliser

① T^*G

$$L\,dt = p_I\,dq^I - A^a\left(J_a(q,p) - \phi_a\right), \qquad I, a = 1, \cdots, \dim G$$

$$J_a(q,p) = J_a^{\ I}(q)\,p_I$$

Solution of constraints : $p_I = \theta_I^{\ a}(q)\,\phi_a$

$\theta_I^{\ a}(q)$: inverse of $J_a^{\ I}(q)$

Maurer-Cartan form $\theta(q) = \theta_I^{\ a}(q)\,dq^I\,X_a = g^{-1}dg$

$$\Rightarrow \quad L\,dt = \theta_I^{\ a}(q)\,dq^I\,\phi_a = \langle \phi, g^{-1}dg \rangle$$

② $\mathbb{R}^{2N}/\mathbb{Z}_2$ (Dual pair correspondence)

$(G_1, G_2) \subset Sp(2N, \mathbb{R})$

Coadjoint orbit correspondence
(symplectic reduction)

Any reductive group G_1 can be embeded in $Sp(2N, \mathbb{R})$
with a suitable G_2

Foliation of $\mathbb{R}^{2N}/\mathbb{Z}_2$ by $\underbrace{\mathcal{O}_\phi^{G_1}} \times \underbrace{\mathcal{O}_\chi^{G_2}}$

1-to-1

quantization

Decomposition of metaplectic rep into $\pi_\phi^{G_1} \otimes \pi_\chi^{G_2}$

Application to spinning ptcle (work in progress)

Poincaré :

- massive spin s $\qquad \phi = m\tilde{P}^0 + s\tilde{J}^{12}$
- massless spin s $\qquad \phi = E\tilde{P}^+ + s\tilde{J}^{12}$
- continuous spin σ $\qquad \phi = E\tilde{P}^+ + \sigma\tilde{J}^{-1}$

AdS and dS analogues are easy to identify

Various procedures

- Suitable decomposition of $\mathfrak{g}$ ("nonlinear realization")
- T^*G
- Dual pair

Dual pair

"Vector" : $\big(O(p,q),\ Sp(2n,\mathbb{R})\big) \subset Sp(2N,\mathbb{R})$

$N = n(p+q)$

- $n = 1$: scalar
- $n = 2$: symmetric spin s
- $n \geq 3$: mixed sym with $(n-1)$-row Young diagram

"Spinor" :

$so(2,3) \simeq sp(4,\mathbb{R})$	$\big(Sp(4,\mathbb{R}),\ O(n)\big)$
$so(1,4) \simeq sp(1,1)$	$\big(Sp(1,1),\ O^*(2n)\big)$
$so(2,4) \simeq u(2,2)$	$\big(U(2,2),\ U(n)\big)$
$so(1,5) \simeq u^*(4)$	$\big(U^*(4),\ U^*(2n)\big)$
$so(2,6) \simeq so^*(8)$	$\big(O^*(8),\ Sp(n)\big)$

What happens when G is not a spacetime symmetry?

Take a suitable subgroup $H \subset G$ as a spacetime symmetry and interprete G as an EXTENDED symmetry

Ex :
- supersymmetry

  $\rightsquigarrow$ supermultiplet $\Rightarrow$ contains a spin s ptcle
- higher spin symmetry

  ∞-dim symmetry
- colored isometry

  physically exotic but mathematically familiar

Ex : Colored isometry in 3d

AdS_3 symmetry : $so(2,2) \simeq so(1,2) \oplus so(1,2)$

$$so(1,2) \simeq su(1,1)$$

Dress $so(2,2)$ with color symmetry $su(N)$

$$\Rightarrow \quad su(N,N) \simeq \underbrace{\overset{X^a}{su(1,1)}}_{\substack{\text{original}\\ \text{isometry}}} \oplus \underbrace{\overset{T^I}{su(N)}}_{\substack{\text{color}\\ \text{symmetry}}} \oplus \underbrace{\overset{X^{aI}}{su(1,1) \otimes su(N)}}_{\text{extra generators}}$$

Colored Poincaré $\quad su(N,N) \oplus su(N,N)$

$$\Big\downarrow \ \text{Inonu-Wigner contraction}$$

$$su(N,N) \in\!\!\!\!+ \ su(N,N)$$

Take the coadjoint orbit with $\phi = m\tilde{\mathbb{P}}^0$

$: S = \int \langle \phi, g^{-1}dg \rangle \qquad g \in SU(N,N) \ltimes SU(N,N)$

introduce constraint by hand

$= \int \mathrm{Tr}\left[\mathbb{P}\dot{\mathbb{X}} + \Lambda(\mathbb{P}^2 - m\mathbb{I})\right]$

$\mathbb{X}, \mathbb{P} \in su(N,N)$: $2N \times 2N$ traceless Hermitian matrix

Λ : $2N \times 2N$ Hermitian matrix

$\Rightarrow$ disjoint union of $N+1$ semisimple coadjoint orbits of dimension $2N^2$

Massless case : $\mathbb{P}^2 = 0$

$\Rightarrow$ Closure of nilpotent orbit of dimension $2N^2$

it contains sub nilpotent orbits of dimension $4Nk - k^2$

$k = N-1, N-2, \cdots, 0$

Interesting examples of Poisson geometry. Quantization ?

Irreps of $SU(N,N)$ from dual pair correspondence

- Massive case $(U(N,N), U(N))$
- Masslers case $(U(N,N), U(k))$

Future directions

- Symplectic reflection symmetry and deformation
- Relation to BF theory and Poisson σ model
- Higher spin symmetry and particles
- String extension