Class of 2D sigma models $\mathbb{C}P^{n-1}$ ... $\longleftrightarrow$ 2D Gross-Neveu models ("φ^4 theory in 2D").

complicated

easier?

$$X : \Sigma_2^{(z,\bar z)} \longmapsto \mathcal{M}^{(X)}$$

worldsheet

target space

$$S' = \frac{1}{2}\int d^2z \sum_{\alpha=1}^{2} \partial_\alpha X^\mu \partial_\alpha X^\nu \underline{G_{\mu\nu}}(X)$$

Dirichlet energy functional.

metric on $\mathcal{M}^{(X)}$

$\Sigma_2^{(z,\bar z)} = \underline{\mathbb{C}}, \mathbb{C}P^1$, cylinder, 

" $Z = \int \prod dX \, e^{-S'}$ " path integral.

Sigma models (class.) $\leftrightarrow$ Harmonic maps

$\mathcal{M} = \mathbb{C}$

$\Delta X = 0$

Holomorphic maps $\subset$ Harm. maps.

"

Instantons

$$(\Sigma, \gamma) \mapsto S = \frac{1}{2}\int d^2z \sqrt{\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) =$$

inv. w.r.t. $\gamma_{\alpha\beta} \mapsto \Omega \gamma_{\alpha\beta}$.

$$ds^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta = \Omega\, dz\, d\bar{z}$$

$$= \int d^2z\, \partial_z X^\mu \partial_{\bar{z}} X^\nu G_{\mu\nu}(X)$$

In 1D: mechanical particle on M $\quad \mathcal{L} = \frac{m\vec{v}^2}{2}$.

(geodesic flow on M).

Quantization: QM on M

$$H = -\Delta_M$$

Gross-Neveu

$$\mathcal{L} = \sum_{a=1}^{n} \bar{\psi}^a \not{D} \psi^a + \varkappa \sum_{a=1}^{n} \bar{\psi}^a \frac{1+\sigma_3}{2}\psi^a \sum_{b=1}^{n} \bar{\psi}^b \frac{1-\sigma_3}{2}\psi^b$$

$$\psi^a = \begin{pmatrix} U^a \\ \bar{V}^a \end{pmatrix}$$

coupling

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Chiral model.

Poincare-Liouville one-form θ: $d\theta = \Omega$

U(1)-covariant derivative

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu - i A_\mu)$$

ψ^a are commuting variables.

- First-order formulation
- $\mathbb{CP}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^*$ ← $U(1)$ gauge field A_μ (involves chiral symmetry).

GLSM

gauged linear sigma

$$\mathcal{L} = \sum_{a=1}^{n} \left(V^a \bar{D} U^a + \bar{V}^a D \bar{U}^a \right) + x \sum_{a=1}^{n} |U^a|^2 \times \sum_{b=1}^{n} |V^b|^2$$

$\beta\gamma$ system.

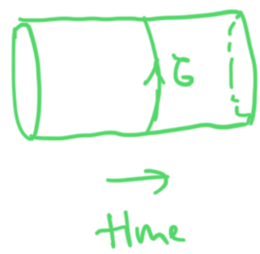
$+ \xi \bar{A} + \bar{\xi} A$

$U^a \mapsto \lambda U^a$

$V^a \mapsto \lambda^{-1} V^a$ $\quad \lambda \in \mathbb{C}^*$

$\bar{D} U^a = x |U|^2 \bar{V}^a$

We gauge $\mathbb{C}^*$.



U^a is a section of some line bundle L over Σ.

V^a " " dual $\quad L' = K \otimes L^{-1}$.

$$(T^*\mathbb{C}^n)^{\text{stab}} /\!/ \mathbb{C}^* \overset{?}{\simeq} T^*\mathbb{CP}^{n-1}$$

↑ complex sympl. red.

$\{U^a\} \neq 0$.

$\partial \bar{U}^a + i A \bar{U}^a$

$\bar{A}: \quad \sum_{a=1}^{n} U^a \bar{D} \bar{U}^a = 0$

$$A = i \frac{\sum U^a \partial \bar{U}^a}{\sum |U^b|^2}$$

[$\varphi \quad \frac{1}{\ }\ \sum^{n} |\bar{D} U^a|^2$ $\quad \mathbb{CP}^{n-1}$ sigma model.]

$$\mathcal{L} = x \frac{a=1}{\cancel{\sum_{b=1}^{n} |U^b|^2}} \mapsto \mathbb{CP} \text{ sigma model}$$

$$\mathcal{L} = \frac{1}{x} G_{a\bar{b}} \partial U^a \bar{\partial} \overline{U^b}, \quad G_{a\bar{b}} = \frac{1}{|U|^2}\left(\delta^{a\bar{b}} - \frac{\overline{U^a} U^b}{|U|^2}\right).$$

$$U^a \mapsto \lambda(z,\bar{z})\, U^a$$

Partial gauge: $\boxed{\sum_{a=1}^{n} |U^a|^2 = 1}$ $\mapsto \mathbb{C}^* \mapsto U(1)$

A different gauge $U^n = 1$.

$$S = \int d^2z\, \mathcal{L} = \int d^2z\, \mathcal{L}_{\mathbb{CP}^{n-1}} + \int \Omega$$

top. term.

Variation of $\mathcal{L}$ w.r.t. $\bar{A}$:

$$\boxed{\sum_{a=1}^{n} V^a U^a = \underline{\zeta} \in \mathbb{Z}}$$

$U^n = 1$ (gauge) $\mapsto V^n = -\sum_{b=1}^{n-1} V^b U^b$

Gauge transformation:

$\bar{\ }\quad \bar{\ }\quad \bar{\partial}\,\Lambda(z,\bar{z}) \qquad \lambda \in \mathbb{C}^*$

$A \mapsto A + i\partial \ln \lambda(z,\bar z)$

$\int d^2 z \, F_{z\bar z}^2$

$F_{z\bar z}$ is not gauge-invariant.

Another (useful) gauge:

$$(\bar A)_{new} = \bar A + i\bar\partial \ln\lambda = 0$$

$$\ln\lambda^{(z,\bar z)} = \frac{i}{\pi}\int d^2 w \, \frac{\bar A(w,\bar w)}{z-w}.$$

$$\boxed{A = \bar A = 0} \longleftrightarrow \underline{U^n = 1}$$

"conformal" gauge

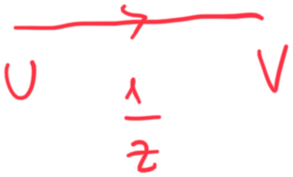
"lightcone" gauge.

Moduli of holomorphic vector bundles over Σ

$\updownarrow$

Flat connections

$\pi_1(\Sigma) \mapsto SU(n)$.



$\varkappa = \frac{1}{R^2}$

$$\bar\partial J = \varkappa\,[J, \bar J]$$ principal chiral model.

SUSY $\mathbb{CP}^{n-1}$ model $\longleftrightarrow$

$$\boxed{S' = S_0 + \varkappa \int d^2 z \, \mathrm{Tr}(J\bar J)}$$

$+ \tilde\varkappa \, \mathrm{Tr}(J^2) + \bar{\tilde\varkappa}\, \mathrm{Tr}(\bar J^2).$

$$J^a(z) J^b(w) = \frac{k\,\delta^{ab}}{(z-w)^2} + \dots$$

$\mathrm{Tr}(AJ^2)$

$J = U \otimes V$

level.

$VU = 0$

$\langle J^a(z) J^b(w) \rangle = 0.$ $\quad J^2 = 0$

$J = U \otimes V - B \otimes C.$

conjecture (LeClair et.al. '2001).

$$\left[\ \beta_x = -\frac{C_2 x^2}{\underline{(1+\frac{1}{2}kx)^2}}\ \right]$$

chiral GN-model

$k=0$ (SUSY)

$C_2 =$ Adj. rep. Casimir of symmetry group G.

$SU(n)$: $C_2 = n$.

SUSY $\mathbb{CP}^{n-1}$ model:

β-function one-loop exact.

$$\mathcal{L}_0 = \sum V^a \bar{D} U^a + B^a \bar{D} C^a$$

$(\beta\gamma)$ $\quad$ (bc)

symplectic bosons.

* SO/Sp sigma models. $\mapsto$ $J^2 \neq 0$

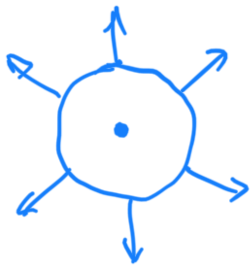
$J^3 = 0.$

Particle on $\mathbb{CP}^{n-1}$

$$\mathcal{L} = \frac{1}{2} \sum G_{ij} \dot{x}^i \dot{x}^j + \xi \underline{\sum A_i \dot{x}^i}$$

$J^2 = 0 \mapsto J^2 = \xi \mathbb{1}_n$.

$\mathbb{CP}^1$



Quantize

$$H = -\Delta_G \qquad (A_i = 0)$$

$$H = -\Delta_G^{(A)} \qquad \text{(Bochner Laplacian).}$$

$\mathcal{N}=2$ SUSY : $H = -\not{D}^2$ on $\mathbb{CP}^{n-1}$.

$$\sum V^a U^a = 0 \xrightarrow[\text{quant.}]{} \sum U^a \frac{\partial \psi}{\partial U^a} = k, \quad \sum \bar{U}^a \frac{\partial \psi}{\partial \bar{U}^a} = -k$$

wave function is homogeneous in $\{U^a\}$.

$$U^a \mapsto e^{i\phi} U^a$$

$$\psi(e^{i\phi} U^a) = e^{i\phi k} \psi(U^a)$$

upcoming joint work with A. Smilga

Phase space = (super) Quiver variety Q

of GN model

If $Q = T^*\mathcal{M} \Leftrightarrow$ sigma model on $\mathcal{M}$.

Nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$.

$\updownarrow$

{ Quiver varieties. GN side

resolution

$\pi: \mathcal{M} \to N_0$

$T^*(G/H)$ sigma model side.

↑ target space.

4D ↔ 2D $\quad \dfrac{SL(2,\mathbb{R})}{SO(2)} \qquad \dfrac{SU(1,n-1)}{SU(n-1)\times U(1)}$ $\quad n = 2 + \#$ vector fields.

Breitenlohner & Maison.

↑ $n=2$.

G/H

ball in $\mathbb{C}^2$ $\quad |z_1|^2 + |z_2|^2 < 1$.

$$V_{int.} = |U|^2|V|^2 + |U|^4 + |V|^4$$

$$SU_n: \quad J^2 = 0 \leftrightarrow J^2 = a\,\mathbb{1}_n$$

$so, sp: \bar{J} = 0 \mapsto J = a\,\mathbb{1}_n$

$SO(n): \dim \mathcal{U}^{min} = 2(n-3)$

$$T^* \frac{SO(n)}{SO(2)\times SO(n-2)} = 2(n-2).$$

$$\mathcal{L} \mapsto \mathcal{L} + \cancel{\bar{\xi}\bar{A}} + \bar{\xi} A$$

$U(1)$

"Lorentz"