Noncommutative Quantum Field Theory - A Viable Approach to Quantum Gravity?

Albert Much

University of Leipzig,
Institute for Theoretical Physics,
D-04081 Leipzig,
Germany

Seminar IFS

Motivation

Does NC Geometry award us with a theory of quantum gravity?

QFT in Curved-Spacetimes is a zero order approximation to QG

$\Rightarrow$ Fuse NCG with QFT in Curved-Spacetimes to obtain first order approximation.

Intro, What

- **Generalize NQFT** in Minkowski to **curved spacetime** rigorously

- Prove that it complies with the **equivalence principle**

Intro, Why

- QFT in curved spacetimes provided various achievements (e.g. black hole entropy, quantum inequalities, descriptions of early universe, particle interpretation, ...)
- Any theory of QG must contain QFT in CST as a limiting case
- Supply a proof as to the **connection to quantum gravity**

NC Generalization

Goal: Generalize Star Product (or Rieffel product)

$$(f \star_\theta g)(z) = \lim_{\epsilon \to 0} \iint \chi(\epsilon x, \epsilon y)\, f(z + \Theta x)\, g(z + y)\, e^{-i\, x \cdot y} d^4 x\, d^4 y$$

from flat to curved manifolds!

Technical problem: In General Spacetimes there are NO translations

Idea in Fröb, Much [JMP21] in case of de Sitter: Embed the spacetime in a higher dimensional flat Minkowski where translations exist

This idea is further **developed to all globally hyperbolic spacetimes**, [AM, A Deformation Quantization for Non-Flat Spacetimes and Applications to QFT '21]

Embedding in GHST

Theorem (Sanchez, Müller 11)

Let (M, g) be a globally hyperbolic spacetime. Then, it admits an isometric embedding in $\mathbb{L}^N$

$\exists\ F : M \to \mathbb{L}^N$, with local coordinates $X^A = (X^\mu, X^a)$

$$\sum_{A=0}^{N} \frac{\partial X^A}{\partial x^\mu} \frac{\partial X_A}{\partial x^\nu} = g_{\mu\nu}.$$

Existence of $F \equiv$ existence of solutions for Diff. equations

$$X^A = X^A(x^\mu).$$

Embeddings

Example

For example, in the case of embedding the unit sphere $\mathbb{S}^2$ into $\mathbb{R}^3$, the differentiable map $F : \mathbb{R}^3 \to \mathbb{R}$ is

$$F(X) = X_1^2 + X_2^2 + X_3^2 - 1,$$

or for the N-dimensional de Sitter $d\mathbb{S}^N$ a map $F : \mathbb{R}^{N+1} \mapsto \mathbb{R}$ is

$$F(X) = -X_0^2 + X_i^2 + X_N^2 - H^{-2},$$

with index $i = 1, \cdots, N-1$ and where H is the Hubble parameter.

Moving in the Embedded Spacetime

Theorem

Let $\mathcal{M}$ be an embedded submanifold of $\mathbb{R}^N$. Then, the tangent space $T_X\mathcal{M}$ is given as the kernel of the differential of any local defining function F at X,

$$T_X\mathcal{M} = ker\, DF(X).$$

Example

$\mathbb{R}^{N-1} = \{X \in \mathbb{R}^N : X^N = 0\}$ (trivially) embedded into $\mathbb{R}^N$, where $F(X) = X^N$. The differential of F is $DF(X)[V] = V^A \nabla_A F$. The tangent space $T_X\mathbb{R}^N$ is the set of all vectors V where $DF(X)[V] = 0$. Since the gradient of F vanishes if $A \neq N$ and is 1 if $A = N$ the kernel of $DF(X)$ consists of all $V \in \mathbb{R}^N$ such that $V^N = 0$, i.e.

$$T_X\mathbb{R}^{N-1} = \ker DF(X) = \mathbb{R}^{N-1}.$$

Example

For the unit sphere $\mathbb{S}^{N-1} = \{X \in \mathbb{R}^N : X_A X^A = 1, A = 1, \cdots, N\}$, we have $DF(X)[V] = X_A V^A$and the tangent spaces $T_X\mathbb{S}^{N-1}$ for $X \in \mathbb{S}^{N-1}$ are the vectors V s.t. $DF(X)[V] = 0$, i.e.

$$T_X\mathbb{S}^{N-1} = \ker\, DF(X) = \{V \in \mathbb{R}^N : X_A V^A = 0\}.$$

Orthogonal Projector

Lemma

The orthogonal projector $P_Z : \mathbb{R}^N \mapsto T_Z\mathcal{M}$ $Z \in \mathcal{M}$*, which is a smooth map, can be represented by a matrix* $P \in \mathbb{R}^{N\times N}$

$$Proj_Z(V^A) = P^A_C(Z)\, V^C.$$

which is uniquely determined by the conditions

$$P(Z) = P(Z)^2 = P(Z)^T$$

for $Z \in M$ *and* $V \in T_Z M$

$$P(Z)V = V.$$

Example

For the $N-1$ dimensional (unit) sphere embedded into a one dimensional higher Euclidean space the orthogonal projector is

$$(\mathrm{Proj}_X)_A{}^B = \delta_A{}^B - X_A X^B,$$

where $\mathrm{Proj}_X : \mathbb{R}^N \to T_X \mathbb{S}^{N-1}$,

$$\begin{aligned}(X, \mathrm{Proj}_X(V))_X &= X^A(\delta_A{}^B - X_A X^B) V_B \\ &= X^A V_A - \underbrace{X^2}_{=1} X^B V_B = 0.\end{aligned}$$

Gradient

Proposition

Let $(\mathcal{M}, g)$ be a (Pseudo-)Riemannian submanifold of $(\mathbb{R}^N, (\cdot, \cdot)))$ and let $f : \mathcal{M} \to \mathbb{R}$ be a smooth function. The (Pseudo-)Riemannian gradient of f is defined by

$$\nabla_A f(X) = Proj_X(\partial_A \bar{f}(X))$$

where $\bar{f}$ is any smooth extension of f to a neighborhood of $\mathcal{M}$ in $\mathbb{R}^N$ and ∂ denotes the (Pseudo-)Euclidean gradient w.r.t. the coordinate X.

Retraction

Next, we introduce the concept of retraction at the point $x \in \mathcal{M}$.

Definition

A retraction on a manifold $\mathcal{M}$ is a smooth map R from the tangent bundle $T\mathcal{M}$ onto the manifold $\mathcal{M}$

$$R : T\mathcal{M} \to \mathcal{M} : (x, v) \mapsto R_x(v)$$

such that each curve $c(t) = R_x(tv)$ satisfies $c(0) = x$ and $c'(0) = v$.

Definition

A second-order retraction R on a (Pseudo-)Riemannian manifold $\mathcal{M}$ is a retraction such that, for all $x \in \mathcal{M}$ and all $v \in T_x\mathcal{M}$, the curve $c(t) = R_x(tv)$ has zero acceleration at $t = 0$, that is, $c''(0) = 0$.

Examples

Example

The retraction on a linear manifold $\mathbb{R}^N$ is a translation

$$R_X(\xi) = x + \xi,$$

where $x \in \mathbb{R}^N$ and $\xi \in T_x\mathbb{R}^N$.

Example

Let $\mathcal{M} = \mathbb{S}^{n-1}$ be the $(n-1)$-dimensional sphere, then a retraction is specified by

$$R_X(\xi) = \frac{X + \xi}{\|X + \xi\|} = \frac{X + \xi}{\sqrt{1 + \|\xi\|^2}}$$

where $X \in \mathbb{S}^{n-1}$ and $\xi \in T_X\mathbb{S}^{n-1}$.

Using the gradients, hessians, projectors and the second-order retraction we have a Taylor expansion on curves given by,

$$f(R_x(v)) = f(x) + (v, \nabla f(x))_x + \frac{1}{2}(v, \nabla_v \nabla f(x))_x + \mathcal{O}(v^3).$$

Poisson Bivector

Definition

A *Poisson bivector* on a smooth manifold $\mathcal{M}$ is a smooth bivector field $\pi \in \Gamma^\infty(\Lambda^2(T\mathcal{M}))$, i.e. π is a smooth skew-symmetric tensor, satisfying

$$\pi^{AB}\partial_A\pi^{CD} + \pi^{AC}\partial_A\pi^{DB} + \pi^{AD}\partial_A\pi^{BC} = 0.$$

Choosing local coordinates (U, X), any Poisson bivector is given by

$$\pi_{|U} = \frac{1}{2}\sum_{A,B}\pi^{AB}\frac{\partial}{\partial X^A}\wedge\frac{\partial}{\partial X^B}.$$

The connection between the Poisson bracket and the Poisson bivector is

$$\{f, g\} = \sum_{A,B}\pi^{AB}\partial_A f\,\partial_B g.$$

The Curved Star Product

Definition

Let the matrix $\Theta := \theta\,\pi$, where $\theta \in \mathbb{R}$. Then, the generalized Rieffel product of two functions $(f, g) \in \mathcal{D}^m_{\rho,\delta}(\mathcal{M})$ is defined as

$$(f \star_\theta g)(z) \equiv \lim_{\epsilon \to 0} \iint \chi_\epsilon(X, Y)\, f(R_Z(\Theta X))\, g(R_Z(Y))\; e^{-i(X,Y)_z} dX\, dY$$

where Z is the embedding point corresponding to z, $X, Y \in T_Z\mathcal{M}$.

Properties of the Non-flat Star Product

Proposition

For functions $(f, g) \in \mathcal{D}^m_{\rho,\delta}(\mathcal{M})$ the generalized Rieffel product is well-defined and satisfies the following properties

- *Unital,*

$$1 \star_\theta f = f \star_\theta 1 = f$$

- *The commutative limit,*

$$\lim_{\theta \to 0} (f \star_\theta g)(z) = (f \cdot g)(z),$$

- *The flat limit: Retractions are translations with constant Poisson bivector such that the generalized Rieffel product turns to the standard star product.*

Associativity

Theorem

Let the Poisson bivector $\pi \in \Gamma^\infty(\Lambda^2(T_Z\mathcal{M}))$ *fulfill*

$$\nabla_U \pi^{CD}(Z) = 0$$

for all $U \in T_Z\mathcal{M}$ *and* $Z \in \mathcal{M}$*. Then, the generalized Rieffel product is associative up to second order in* θ*, and it is explicitly given by*

$$(f \star_\theta g)(z) = fg - i\Theta^{AB}\,\partial_A f\,\partial_B g - \frac{1}{2}\Theta^{AC}\Theta^{BD}\partial_A \nabla_B f\,\partial_C \nabla_D g,$$

for functions $(f, g) \in \mathcal{D}^m_{\rho,\delta}(\mathcal{M})$*.*

NC Structures

Proposition

The generalized Rieffel product is Poisson compatible,

$$\frac{i}{2}(f \star_\theta g - g \star_\theta f) = \theta \{f, g\}_\pi.$$

Hence, the noncommutativity of the coordinates $\{X^A, X^B\} = \pi^{AB}$, follows explicitly from the deformation, i.e.

$$[X^A, X^B]_\theta = -2i\theta\pi^{AB}$$

For the flat case,

$$[x^\mu, x^\nu]_\theta = -2i\theta\pi^{\mu\nu},$$

and for the two-sphere

$$[X^A, X^B]_\theta = -2i\theta\, \varepsilon^{AB}{}_C X^C.$$

Deformed Product for Two Different Points

The main motivation for this extension is the application of deformations to QFT (e.g. two-point function).

Definition

Let the matrix $\Theta := \theta\,\pi$, where $\theta \in \mathbb{R}$. Then, the generalized Rieffel product of two functions $(f, g) \in \mathcal{D}^m_{\rho,\delta}(\mathcal{M})$ at two different points is defined as

$$f(z_1) \star_\theta g(z_2) \equiv \lim_{\epsilon \to 0} \iint \chi_\epsilon(X, Y)\, f(R_{Z_1}(\Theta\, \delta(Z_1, Z_2) X))\, g(R_{Z_2}(Y)) \times\ e^{-i\,(X,Y)_{z_2}}$$

where Z_1 and Z_2 are the embedding point corresponding to z_1 and z_2, $X, Y \in T_{Z_2}\mathcal{M}$. Moreover, ${\delta_B}^{A'}(Z_1, Z_2)$ is the operator of geodesic transport from $T_{Z_2}\mathcal{M}$ to $T_{Z_1}\mathcal{M}$.

Explicit Star Product

Proposition

The deformed product for two different points is given up to second order in the deformation parameter as,

$$f(z_1) \star_\theta g(z_2) = f(z_1) g(z_2) - i\Theta^{AB'}(Z_1, Z_2)\, \partial_A f(Z_1)\, \partial_{B'} g(Z_2) \\ - \Theta^{ABC'D'} \partial_A \nabla_B f(Z_1)\, \partial_{C'} \nabla_{D'} g(Z_2) + \mathcal{O}(\Theta^3),$$

where we defined

$$\Theta^{AB'}(Z_1, Z_2) := \Theta^{AB}(Z_1)\, \delta_B{}^{B'}(Z_1, Z_2),$$

$$\Theta^{ABC'D'}(Z_1, Z_2) := \frac{1}{4}\left(\Theta^{AC}\,\Theta^{BD} + \Theta^{AD}\,\Theta^{BC}\right) \delta_D{}^{D'}(Z_1, Z_2)\, \delta_C{}^{C'}(Z_1, Z_2).$$

$\Rightarrow$ Next, we apply the developed methods to QFT in Curved Spacetimes

Intro QFTCST

QFTs are rigorously constructed for Globally Hyperbolic spacetimes

Advantages: Exist direction of a time, well-posed Cauchy problem

Disadvantages: No preferred State

(GNS) For a given state (positive linear functional) ω over the (unital) $*$-algebra $\mathscr{A}$, one obtains a quadruple $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Psi_\omega)$. Field operators are given by

$$\phi_\omega(F) = \pi_\omega(\phi(f)) : D_\omega \to \mathcal{H}_\omega$$

Then the n-point functions are given by

$$\omega_n(\phi(F_1)\cdots\phi(F_n)) = \langle \Psi_\omega | \, \pi_\omega(\phi(F_1))\cdots\pi_\omega(\phi(F_n)) \, \Psi_\omega \rangle$$

Hadmard States

Preferred states: Hadamard States (resemble singularity structure Minkowski)

In a convex neighborhood C of (M, g) the Hadamard parametrix is

$$H_\epsilon(x, y) = \frac{u(x, y)}{\sigma_\epsilon^2(x, y)} + v(x, y) \log\left(\frac{\sigma_\epsilon^2(x, y)}{\lambda^2}\right)$$

where $\sigma^2(x, y)$ is the geodesic distance (the Synge function), T is any local time coordinate increasing towards the future, $\lambda > 0$ a length scale and

$$\sigma_\epsilon^2(x, y) \stackrel{\text{def}}{=} \sigma^2(x, y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2 \, ,$$

Application to QFT in GHST

Definition

For a $*$-algebra $\mathscr{A} = \mathscr{A}(M, g)$ defined on a globally hyperbolic spacetime (M, g) generated by Klein-Gordon fields the deformed 2-point function is defined as follows

$$\omega_2^{\Theta}(\phi(F_1)\phi(F_2)) := \langle \Psi_\omega | \, \pi_\omega(\phi(F_1)) \star_\theta \pi_\omega(\phi(F_2)) \, \Psi_\omega \rangle$$

$$= \int F_1(x_1) \star_\theta F_2(x_2) \, \langle \Psi_\omega | \, \pi_\omega(\phi(x_1))\pi_\omega(\phi(x_2)) \, \Psi_\omega \rangle \, d\mathrm{vol}_g(x_1) \, d\mathrm{vol}_g(x_2)$$

Does the Deformation make Sense?

Definition

If $u \in \mathcal{D}'(\mathbb{R}^n)$, a pair $(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ is a *regular direction* for u if $\exists$ constants C_N, $N \in \mathbb{N}$ so that

$$\left| \hat{\phi u}(k) \right| < \frac{C_N}{1+|k|^N}, \qquad \forall k \in V, \quad \phi \in C_0^\infty(\mathbb{R}^n)$$

Definition (Wave front set)

The *wavefront* set of u is defined to be

$WF(u) = \{(x,k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) : (x,k)\,\text{is not } \textit{a regular direction for } \text{u}\}.$

Useful properties of the WF set

- If $f \in C_0^\infty$ it has an empty wavefront set $WF(f) = \emptyset$.
- $WF(\alpha u + \beta v) \subset WF(u) \cup WF(v)$ for $u, v \in \mathcal{D}'(\mathbb{R}^n)$, $\alpha, \beta \in \mathbb{C}$.
- If P is any differential operator with smooth coefficients, then

$$WF(Pu) \subset WF(u)$$

Let $\mathcal{N}$ be the bundle of nonzero null covectors on M:

$$\mathcal{N} = \{(x, \xi) \in T^*M : \xi \text{ a non-zero null at } p\}.$$

$$\mathcal{N}^\pm = \{(x, \xi) \in \mathcal{N} : \xi \text{ is future}(+)/\text{past}(-)\text{directed}\}.$$

WF set and QFT

Definition

A state ω obeys the **Microlocal Spectrum Condition** (μSC) if

$$WF(\omega_2) \subset \mathcal{N}^+ \times \mathcal{N}^-.$$

Theorem (Radzikowski 96)

The μSC is equivalent to the ***Hadamard condition****.*

Theorem

If ω and ω' obey the μSC then

$$\omega_2 - \omega_2' \in C^\infty(M \times M),$$

i.e. the μSC determines an equivalence of class of states under equality of the two-point functions modulo C^∞.

Micro-local condition in the deformed Setting

Theorem

Let the state ω obey the microlocal spectrum condition. Then, the deformed state ω_Θ obeys the microlocal spectrum condition as well, i.e.

$$WF(\omega_2^\Theta) \subset \mathcal{N}^+ \times \mathcal{N}^-.$$

Proof.

$$\begin{aligned}
\omega_2^\Theta(X_1, X_2) &= P^\Theta \omega_2(X_1, X_2) \\
&= \omega_2(X_1, X_2) - i\,\partial_A \partial_{B'}(\Theta^{AB'}(X_1, X_2)\omega_2(X_1, X_2)) \\
&\qquad - \nabla_A \partial_B \nabla_{C'} \partial_{D'}(\Theta^{ABC'D'}(X_1, X_2)\omega_2(X_1, X_2)),
\end{aligned}$$

where P^Θ is a fourth-order differential operator with smooth coefficients, that depend on the Poisson bivector, the orthogonal projection and the parallel transport.

□

Corollary

*The deformed two-point function ω_2^{Θ} is **Hadamard**.*

$\Longrightarrow$ Deformations **physically meaningful** since they satisfy the **equivalence principle**

Uniqueness of Embeddings

Theorem

Let the deformed states ω_2^{θ} *and* $\omega_2'^{\theta}$ *be defined using either two different isomorphic embeddings or/and retractions. Then, the deformed two-point functions* ω_2^{Θ} *and* $\omega_2'^{\Theta}$ *have the same wavefront set, i.e.*

$$WF(\omega_2^{\Theta}) = WF(\omega_2'^{\Theta}).$$

Proof.

Due to the smoothness of the embedding, the deformed two-point functions ω_2^{Θ} and $\omega_2'^{\Theta}$ obey the μSC and according to Theorem 24, their difference is smooth. Hence, by Property 1 of the WF set we have

$$WF(\omega_2^{\Theta} - \omega_2'^{\Theta}) = 0.$$

□

Conclusions and Outlook

- QFT in noncommutative (or quantized) curved spacetimes agrees with the equivalence principle
- Deformation can be extended to n-point functions
- Rigorous Framework to examine achievements in curved spacetime with NC component, e.g. Hawking effect, Quantum energy inequalities, Entropies (Joint work with H. Grosse, Rainer Verch), semi-classical effects
- Predict testible quantum gravitational effects

Thank you for your Attention!