Bargaining via the Weber-Fechner law

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Outline

- Concept of Pareto optimality
- Bargaining
- Nash general solution
- Weber-Fechner (WF) law in bargaining

Pareto-optimality

$$\begin{cases} u_1(x_1, x_2) \to max \\ u_2(x_1, x_2) \to max \\ (x_1, x_2) \in D \end{cases}$$

How can one define reasonably simultaneous maximization of two functions.

Pareto-optimality

$$\begin{cases} u_1(x_1, x_2) \to max \\ u_2(x_1, x_2) \to max \\ (x_1, x_2) \in D \end{cases}$$

Pareto-optimal (PO)

$$(\hat{x}_1, \hat{x}_2)\ is\ PO \Leftrightarrow \left[\nexists (x_1, x_2)\ s.t. \begin{cases} u_1(x_1, x_2) > u_1(\hat{x}_1, \hat{x}_2) \\ u_2(x_1, x_2) \geq u_2(\hat{x}_1, \hat{x}_2) \end{cases} or \begin{cases} u_1(x_1, x_2) \geq u_1(\hat{x}_1, \hat{x}_2) \\ u_2(x_1, x_2) > u_2(\hat{x}_1, \hat{x}_2) \end{cases} \right]$$

Pareto-optimality

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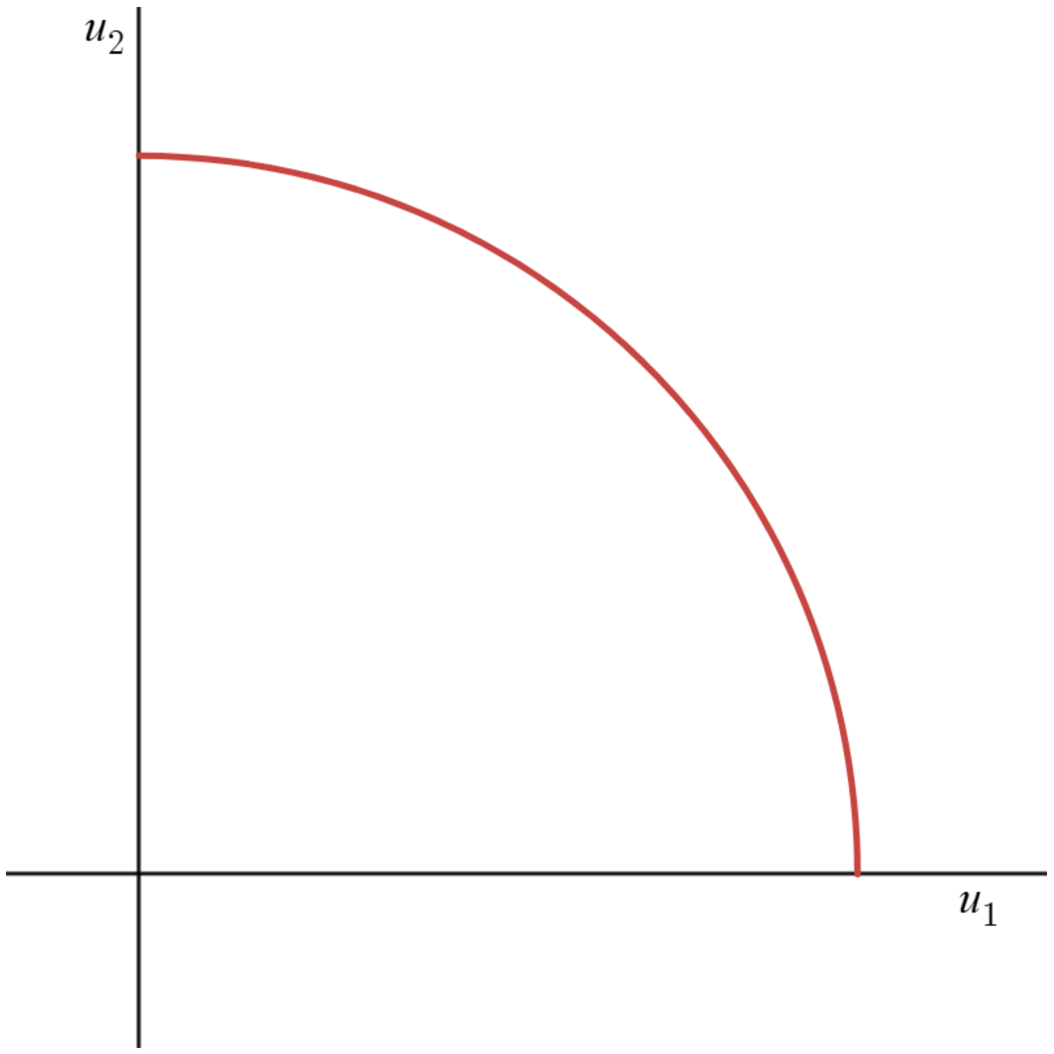
In words, the point is PO if there we **cannot make better in one goal without losing in the other**.

All PO points constitute Pareto-optimal frontier (PF)

Pareto-optimality

- Limited resources (bounded domain (D))
- Two or more **conflicting;** quantifiable goals (i.e. two or more objective functions)
- (Generally) two or more variables

A typical Pareto-frontier

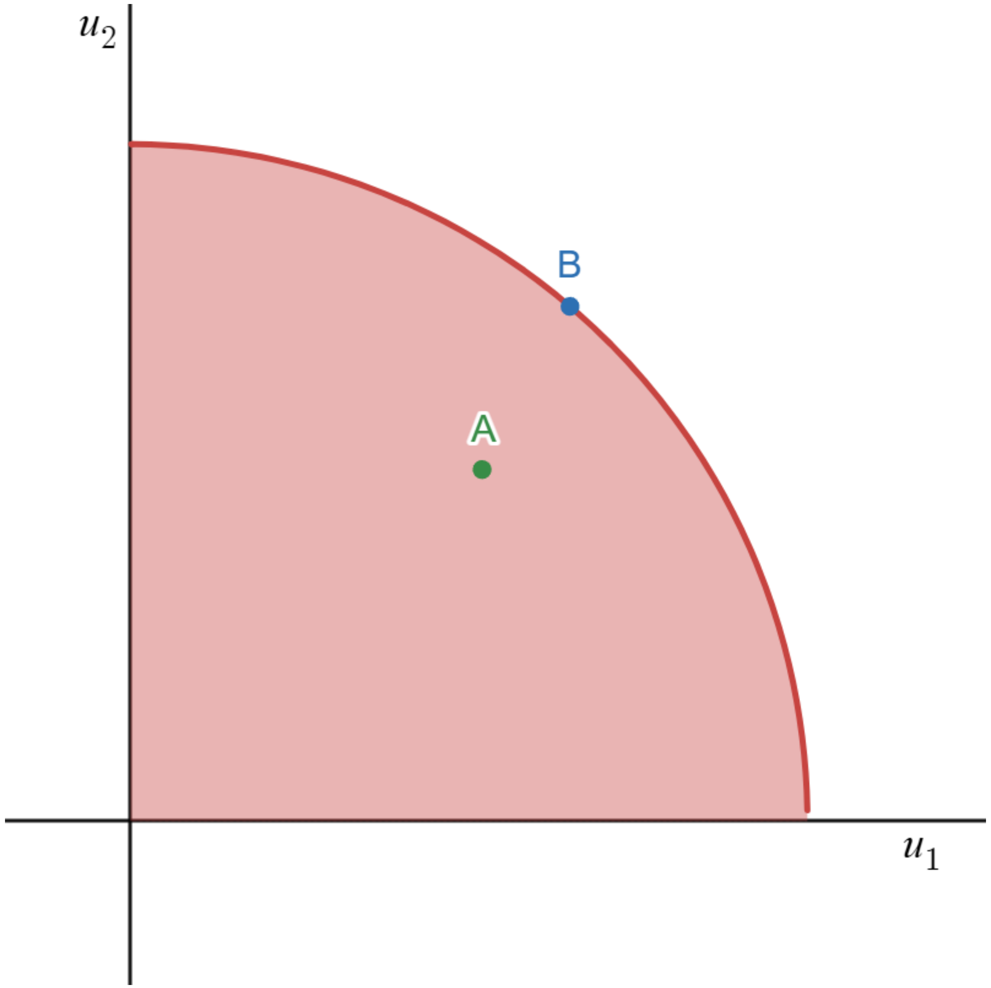


Note that it is given in u-s, not in x-s

This way is visually more appealing.

Each point on red curve is in correspondence with one PO point (x_1, x_2).

A typical Pareto-frontier



No values of objective functions are feasible out of the shaded region.

It is reasonable to expect that $B \succ A$

Pareto-frontier

Typically PF is drawn by solving the following problem for all feasible $\boldsymbol{c}$-s.

$$\begin{cases} u_1(x_1, x_2) \to max \\ u_2(x_1, x_2) = c \\ (x_1, x_2) \in D \end{cases}$$

Pareto-optimality

Was initially proposed by Vilfredo Pareto for income distribution study.

It has, however, found uses in many fields.

- General microeconomics
- Political negotiations and social choice situations.
- Game theory
- Any multi-objective optimization problem.
- **Bargaining**

Bargaining

- Two players
- Negotiate on fair division of given amount
- If they come to any agreement, the split follows
- If they fail to agree, they both takes “defection point” money.

$(d = (d_1, d_2) = (\overline{u}_1, \overline{u}_2))$

Bargaining

Each player has its utility function based on the money each gets.

Generally

- $u_1(x_1, x_2); u_2(x_1, x_2)$
- $x_2 = A - x_1;$
- $x_1; x_2 \geq 0$

- No agreement $\Rightarrow \mathrm{d} = (d_1, d_2)$

- S – domain of values of utility functions.

Bargaining solution

Bargaining solution is and algorithm ψ such that

$$\psi(S, d) \in S$$

The solution is deemed to be "good" if

$$\psi(S, d) \in PF$$

Nash axioms

- Symmetry (**S**)

  if (S, d) is symmetric , then $\psi_1(S, d) = \psi_2(S, d)$

- Scale transformation invariance (**SI**)

  For any $a > 0, b$;

  $$\psi(aS + b, ad + b) = a\psi(S, d) + b$$

Nash axioms

- Pareto optimality (**PO**)

$$\psi(S, d) \in PF$$

- Independence of irrelevant alternatives (**IIA**)

If $S \subset T$ and $\psi(T, d) \in S$, then $\psi(T, d) = \psi(S, d)$

Nash solution

Nash solution

$$\begin{cases} (u_1 - d_1)(u_2 - d_2) \to max \\ (u_1, u_2) \in PF \end{cases}$$

Nash non-symmetric solution

$$\begin{cases} (u_1 - d_1)^{w_1}(u_2 - d_2)^{w_2} \to max \\ (u_1, u_2) \in PF \end{cases}$$

Weber-Fechner law

Weber-Fechner’s law is prominent result of psychophysics.

In simple terms - the change of stimuli to be perceived it must be significant (proportional to initial stimulus).

Weber-Fechner law

Weber's law (1834)

$$\frac{|\Delta S|}{S} \geq k$$

Fechner's addition (1860) - k is different for each individual and for different physical quantities.

Bargaining problem with Weber's law

$$\begin{cases} u_1 \to max \\ \dfrac{|f_1(u_1 + \Delta u_1) - f_1(u_1)|}{f_1(u_1)} \geq k_1 \\ \dfrac{|f_2(u_1 + \Delta u_1) - f_2(u_1)|}{f_2(u_1)} \leq k_2 \\ u_2 = h(u_1) \Leftrightarrow (u_1, u_2) \in PF \\ d = (0,0) \end{cases}$$

Bargaining problem with Weber's law

A simple version would be $(f_1(u_1) = u_1, f_2(\cdot) = h(\cdot))$

$$\begin{cases} u_1 \rightarrow max \\ \dfrac{|\Delta u_1|}{u_1} \geq k_1 \\ \dfrac{|h(u_1 + \Delta u_1) - h(u_1)|}{h\,(u_1)} \leq k_2 \\ u_2 = h(u_1) \Leftrightarrow (u_1, u_2) \in PF \\ d = (0,0) \end{cases}$$

Bargaining problem with Weber's law

For the **I** player (*)

$$\begin{cases} u_1 \to max \\ \dfrac{|\Delta u_1|}{u_1} \geq k_1 \\ \dfrac{|h(u_1 + \Delta u_1) - h(u_1)|}{h\,(u_1)} \leq k_2 \\ u_2 = h(u_1) \Leftrightarrow (u_1, u_2) \in PF \\ d = (0,0) \end{cases}$$

For the **II** player (with $g = h^{-1}$) (**)

$$\begin{cases} u_2 \to max \\ \dfrac{|\Delta u_2|}{u_1} \geq k_1 \\ \dfrac{|h(u_1 + \Delta u_1) - h(u_1)|}{h\,(u_1)} \leq k_2 \\ u_2 = h(u_1) \Leftrightarrow (u_1, u_2) \in PF \\ d = (0,0) \end{cases}$$

Bargaining problem with Weber's law

Linear approximation (is reasonable when k-s are small)

$$\begin{cases} u_1 \to max \\ \dfrac{du_1}{u_1} \geq k_1 \\ \dfrac{|h'(u_1)|du_1}{h\,(u_1)} \leq k_2 \\ u_2 = h(u_1) \Leftrightarrow (u_1, u_2) \in PF \\ d = (0,0) \end{cases}$$

Bargaining problem with Weber's law

Theorem:

1) If $h(\cdot)$ is decreasing and concave function. And $k_1 = k_2$ are enough small ($\rightarrow 0$), then (*;**) solution coincides with Nash solution.

2) If $k_1 \neq k_2$; $\frac{k_1}{k_2} = c$ and $k_1; k_2 \rightarrow 0$; then (*;**) solution coincides with non-symmetric Nash solution.

Bargaining problem with Weber's law

In general solving (*,**) for non-small Weber constants means defining 2 overlapping intervals

$$[u_1^*, u_1^* + \Delta u_1^*]; \;\; [u_2^*, u_2^* + \Delta u_2^*]$$

It can be shown, that

$$h(u_1^* + \Delta u_1^*) < u_2^* < h(u_1^*)$$
$$g(u_2^* + \Delta u_2^*) < u_1^* < g(u_2^*)$$

Bargaining problem with WF law

- So in some sense **Nash solution** can be interpreted as a **special case of Weber's solution**.

- And choosing to concede (i.e. not to commit the last step), open way to iterative procedure.

Ultimatum game

- Two player: **I** and **II**.
- Amount of money to divide: **X**$.

- **I** player is the **proposer**, i.e. he proposes a split $(0 \leq x \leq X)$

$$[\underbrace{x}_{\boldsymbol{I}}\,;\underbrace{X - x}_{\boldsymbol{II}}]$$

- **II** player either agrees or disagrees to split, in which case they both get nothing.

Ultimatum game

- **I** has continuum of options: $\boldsymbol{x} \in [0, X]$
- **II** has only two options: accept or reject.

- Nash sub-game perfect equilibrium states, that ***any non-zero choice should be accepted*** by **II**. So, **I** won't have incentive to propose anything larger than smallest possible amount.

- However, there is huge body of evidence that proposers tend to be more inclined to propose split close to $x = \frac{X}{2}$ ("*fair splits*"). And responders tend to reject small enough ("*unfair*") splits.

Ultimatum game

So here, roles are assigned to each player. And these roles are not switched.

In that, we assume that the last step can still be taken.

i.e.

$$h(u_1^* + \Delta u_1^*)$$

Is what should be chosen.

For a specific form of utility (namely Bernoulli logarithmic utility) most of empirical finding can be addressed.

$$u_i = \ln(x_i + \gamma_i)$$

With γ_i being initial wealth.

Ultimatum game

We were able to show that

- $\overline{x}_1 > \frac{X}{2}$
- In symmetric case, the richer the agents the fairer is their split
- In non-symmetric case, richer agent can behave both like greedy person and philanthropist.

Thank you