# 4D gravity via triples of 2-forms

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# Main message

There exists a very powerful, both geometrically and computationally, formalism for 4D GR

Encodes the spacetime metric into a triple of 2-forms (satisfying some simple algebraic relation)

This formalism is far from being new, it was discovered by Jerzy Plebanski in 1978

Many aspects of this formalism are well-known and standard in the mathematical literature on 4D geometry

The purpose of this talk is to explain you the geometry behind this formalism

And also explain in what sense it is (in my view) superior to the usual metric one

At a very basic level, we are talking here about a convenient and useful choice of variables for GR

# Motivations

Gravity = Geometry

GR is the dynamical theory of the spacetime metric

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda) = 8\pi G T_{\mu\nu}$$

It is now understood that vacuum Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$

Are the unique diffeomorphism invariant second-order in derivatives equations that can be written down (in 4D)

Predictions of GR agree with all our observations of the Universe so far - Cosmology, Gravitational Waves

# Difficulties

Nevertheless, GR is a very difficult theory to deal with

Mathematically, Einstein equations are complicated non-linear hyperbolic (mod gauge) second-order PDE's for the metric

Even the statement of well-posedness was only proven in 1952 (Choquet-Bruhat)

# Hidden simplicity

But there are also hints of hidden simplicity

Many parallels with YM theory (or E&M) but both are much simpler than GR

Studying scattering amplitudes one learns that GR=(YM)^2

Also both of these theories have the so-called self-dual sector which is integrable

One of the main achievements of Penrose's twistor program is establishing this integrability

and also linking it to complex analysis and algebraic geometry

All this is simply invisible in the usual metric formulation

# Cartan's formalism of moving frames

A step towards simplification is made by Cartan's frame formalism

Given some "geometric structure" = collection of tensors

one can consider all frames "compatible" with this structure

For example, in the metric case this would be all orthonormal frames

In many cases, what arises is a principal G-bundle of such compatible frames

One calls this type of geometry that of G-structures

# Riemannian case

For (pseudo-) Riemannian case one obtains the SO(n) or SO(1,n-1) principal bundle of oriented orthonormal frames

This leads to a powerful frame formalism

Concretely, a frame is viewed as a map $\theta : \mathbb{R}^n \to T_pM$

Given such a map one can build the all-important soldering form $e : T_\theta FM \to \mathbb{R}^n$ where $FM \xrightarrow{\pi} M$ is the principal frame bundle

$$e(X) = \theta^{-1}(\pi(X))$$

This gives rise to an $\mathbb{R}^n$ valued 1-form in the associated $\mathbb{R}^n$ bundle

# Frame formalism concretely

Concretely $e^I, \quad I = 1, \dots, n$ collection of 1-forms that are declared orthonormal and thus encode the metric

Consider the metric covariant derivative

Then there exists a collection of coefficients $\omega(X)^I{}_J$ such that $\nabla_X e^I = \omega(X)^I{}_J e^J$

This is true because the covariant derivative preserves the metric, and thus the space of its orthonormal 1-forms

Not difficult to show that $\omega(X)^I{}_K \delta^{KJ} \equiv \omega(X)^{IJ}$ is anti-symmetric $\omega^{IJ} = \omega^{[IJ]}$

Thus, components of an so(n) or so(1,n-1) connection

Then $2\nabla_{[X}\nabla_{Y]} e^I = R(X,Y)e^I = (\nabla_X \omega(Y)^I{}_J - \nabla_Y \omega(X)^I{}_J + \omega(X)^I{}_K \omega(Y)^K{}_J - \omega(Y)^I{}_K \omega(X)^K{}_J) e^J$

Thus, can extract Riemann curvature from the curvature of the connection $\omega^I{}_J$

# Differential forms

The power and the miracle of this formalism is that one can forget about the metric

The connection $\omega^I{}_J$ is completely characterised by the projection of the equation $\nabla_\mu e^I_\nu = \omega_\mu{}^J{}_J e^J_\nu$

onto the space of 2-forms

The resulting equation is $de^I = \omega^J{}_J \wedge e^J$

Number of equations here is the number of unknowns $R(\omega)^I{}_J = d\omega^J{}_J + \omega^I{}_K \wedge \omega^K{}_J$

Thus, one can compute the Riemann curvature by working with the soldering 1-forms

The only operations one needs to do is take exterior derivatives, wedge products, and solve linear equations for $\omega^I{}_J$

One never needs the metric or the Levi-Civita connection in this formalism!

Even Einstein equations can be written in this language of forms $\epsilon^{IJKL} R^{JK} \wedge e^L = \Lambda \epsilon^{IJKL} e^J \wedge e^k \wedge e^L$

# Action principle

There is a simple action principle that realises all these ideas - can be called Einstein-Cartan action

$$S[e,\omega] = \int \epsilon^{IJKL}(R^{IJ} \wedge e^K \wedge e^L - \frac{\Lambda}{12} e^I \wedge e^J \wedge e^k \wedge e^L)$$

In this formalism the metric is encoded into the 1-forms $e^I$

No metric appears explicitly!

This formalism is not optional when one wants to couple gravity to spinors

# Difficulties

In spite of this beauty, the frame formalism is still difficult to deal with in explicit calculations with GR

especially those that need perturbation theory

This is a first-order formalism, with only first derivatives in the action

Kinetic term is schematically of the form $e\partial\omega$

There are 16 components of the soldering form, and 24 components of the connection

The mismatch is too big to be solved by just gauge-fixing

The propagator in this theory is complicated

This is similar to the situation in the metric first-order formalism $h_{\mu\nu}\partial\Gamma_{\mu\nu}{}^{\rho}$

10 components of the metric perturbation, 40 components of the Christoffel symbol perturbation

It is this consideration that suggests to look for an even better alternative

Also quest for formalism that would make simplifications of the self-dual sector manifest

# Self-duality

There is an exceptional isomorphism that can be leveraged to produce an even more efficient formalism

$$\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$$

A related fact is that the space of 2-forms in 4D splits

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

In particular, this means that the spin connection splits $\omega = \omega^+ + \omega^-$

Both $\omega^\pm$ are SO(3) connections

These observations lead to a powerful "chiral" formalism for GR that keeps only one of the two halves of the spin connection

# Geometry of 2-forms in 4D

Given a 4D Riemannian manifold (M,g), Hodge star defines $*: \Lambda^2 \to \Lambda^2$ and provides the decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

Remark: the wedge product metric on $\Lambda^2$ restricted to $\Lambda^+$ is definite

In the opposite direction, the knowledge of Hodge star is equivalent to the knowledge of the conformal metric [g]

Can encode the Hodge star into a choice of a rank 3 subbundle $\Lambda^+$ of $\Lambda^2$ such that the restriction of the wedge product metric to $\Lambda^+$ is definite

$\Lambda^+ \subset \Lambda^2$ such that

$\Lambda^+ \wedge \Lambda^+$ definite $\quad \Leftrightarrow \quad [g]$

Group coset isomorphism

$$\mathrm{SO}(3,3)/\mathrm{SO}(3) \times \mathrm{SO}(3) = \mathrm{SL}(4)/\mathrm{SO}(4)$$

Explicitly $\qquad g(X,Y) \sim i_X B^1 \wedge i_Y B^2 \wedge B^3, \qquad B^{1,2,3} \in \Lambda^+$

Can be checked that $B^{1,2,3}$ are self-dual with respect to this metric

# Encoding a metric into 2-forms

Starting with a metric, we can consider the principal SO(3) bundle of orthonormal frames for $\Lambda^+$

Every such frame can be viewed as a map $\Sigma : \mathbb{R}^3 \to \Lambda^+$

The new feature is that being orthonormal and self-dual they necessarily satisfy the property that

the pull-back of the wedge product metric on $\Lambda^2$ coincides with the metric on $\mathbb{R}^3$

In a basis we can write this as $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

This 2-frame gives rise to an $\mathbb{R}^3$ -valued 2-form, section of the associated bundle

We can now give a description of 4D Riemannian geometry "locally modelled" on this picture

We start with a rank 3 vector bundle $E \to M$ with fibre metric $\delta_{ij}$ and of the same topological type as $\Lambda^+$

We choose a 2-frame $\Sigma : E \to \Lambda^2$ which satisfies the property that the pull-back of the wedge product metric on $\Lambda^2$ coincides with the metric on $\mathbb{R}^3$ i.e., in a basis $\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

Such a 2-frame encodes the metric

18 functions in a triple of 2-forms, minus 5 algebraic constraints = 18-5=13 =10+3

As with usual frames, these 2-forms do not need to be globally defined, they are bundle-valued objects

# Representation theory

Given such a 2-frame structure $\Sigma : E \to \Lambda^2$ possible geometric objects are controlled by the

$\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$ representation theory

One of the two SU(2)'s does not act on $\Sigma$

The other SU(2) acts as SO(3) on $\Lambda^+$ in such a way that $\Sigma : E \to \Lambda^+$ is SO(3) equivariant

All representation theory is controlled by the basic fact that, using the metric defined by $\Sigma$

can "raise" the index of the triple of 2-forms to convert them into a triple of endomorphisms $\Sigma^i \in \mathrm{End}(TM)$

The resulting endomorphisms satisfy the algebra of imaginary quaternions

$$\Sigma^{i\ \alpha}_{\mu}\Sigma^{j\ \nu}_{\alpha} = -\delta^{ij}\delta_{\mu}{}^{\nu} + \epsilon^{ijk}\Sigma^{k\ \nu}_{\mu}$$

# Decomposition of the space of forms

The natural geometric objects in this approach to geometry are E-valued differential forms

We need to understand these objects, and also relate them to more standard tensors

This is all controlled by the simple representation theory of SU(2) x SU(2)

Let us denote the irreducible representations of SU(2) x SU(2) by $S^k_\pm$ $\dim(S^k) = 2k+1$

With the convention that the irreps of the SU(2) that does not act on $\Sigma$ are denoted as $S^k_-$

Then we have the following standard facts $\Lambda^1 = S_+ \otimes S_-$

$$\Lambda^+ = S^2_+$$

Because $E \sim \Lambda^+$ we have

$$E \otimes \Lambda^1 = S^2_+ \otimes S_+ \otimes S_- = (S^3_+ \otimes S_-) \oplus (S_+ \otimes S_-)$$

$$E \otimes \Lambda^2 = S^2_+ \otimes (S^2_+ \oplus S^2_-) = S^4_+ \oplus S^2_+ \oplus \Lambda^0 \oplus S^2_+ \otimes S^2_-$$

# Explicit description of the subspaces

An explicit description of the invariant subspaces in the spaces of E-valued forms arises from two operators

$$J_1 : E \otimes \Lambda^1 \to E \otimes \Lambda^1 \qquad J_1(A)^i_\mu = \epsilon^{ijk} \Sigma^{j\ \alpha}_{\mu} A^k_\alpha \qquad J_1^2 = 2\mathbb{I} + J_1$$

Eigenvalues 2, -1

Eigenspace of eigenvalue 2 spanned by $X^\alpha \Sigma^i_{\alpha\mu}$

$$J_2 : E \otimes \Lambda^2 \to E \otimes \Lambda^2 \qquad J_2(B)^i_{\mu\nu} = \epsilon^{ijk} \Sigma^{j\ \alpha}_{[\mu} B^k_{|\alpha|\nu]} \qquad J_2(J_2 - 2)(J_2 - 1)(J_2 + 1) = 0$$

Explicit parametrisation of eigenspaces

$$\left.\begin{aligned} \Psi^{ij}\Sigma^j &\in (E \otimes \Lambda^2)_5 \\ \epsilon^{ijk}\lambda^j\Sigma^k &\in (E \otimes \Lambda^2)_3 \\ \rho\Sigma^i &\in (E \otimes \Lambda^2)_1 \end{aligned}\right) \quad E \otimes \Lambda^+$$

$$h_{[\mu}{}^{\alpha}\Sigma^i_{|\alpha|\nu]} \in (E \otimes \Lambda^2)_9 \quad E \otimes \Lambda^-$$

$\Psi^{ij}, h_{\mu\nu}$ both tracefree

# Canonical connection

As in the case of frames, there exists a canonical connection in the bundle E after the 2-frame structure is chosen

Proposition: $\nabla_X \Sigma^i \in (E \otimes \Lambda^2)_3$

$\nabla$- Levi-Civita connection of the metric defined by $\Sigma$

To show this, need to show that all other irreducible components in $E \otimes \Lambda^2$ vanish

Because $\nabla$ preserves Hodge $\nabla_X \Sigma^i \in E \otimes \Lambda^+$

So, only need to show that $(\nabla_X \Sigma^i)_{1+5} = 0$

These components are computed as $2\Sigma^{(i|\mu\nu|}\nabla_X \Sigma^{j)}_{\mu\nu} = \nabla_X \Sigma^{i|\mu\nu|}\Sigma^j_{\mu\nu} = 4\nabla_X \delta^{ij} = 0$

Corollary: There exists an E-valued 1-form $A^i_\mu$ such that $\nabla_\mu \Sigma^i_{\rho\sigma} = -\epsilon^{ijk} A^j_\mu \Sigma^k_{\rho\sigma}$

$A^i_\mu$ is a connection in E

This can also be stated by saying that there exists a canonical connection in E such that the total derivative operator

$D = \nabla + A$ is such that $D\Sigma = 0$

Proposition: The connection A is completely determined by the projection to $E \otimes \Lambda^3$ $\quad d\Sigma^i = -\epsilon^{ijk} A^j \wedge \Sigma^k$

Proof: explicit computation, the answer is $A = \dfrac{1}{4}(J_1 - \mathbb{I})(*d\Sigma^i)$

# Curvature

We now take another covariant derivative of $\nabla_\mu \Sigma^i_{\rho\sigma} = -\epsilon^{ijk} A^j_\mu \Sigma^k_{\rho\sigma}$ to get

$$2\nabla_{[\mu}\nabla_{\nu]}\Sigma^i_{\rho\sigma} = R_{\mu\nu[\rho}{}^{\alpha}\Sigma^i_{|\alpha|\sigma]} = -\epsilon^{ijk}F^j_{\mu\nu}\Sigma^k_{\rho\sigma} \quad \Leftrightarrow \quad R_{\mu\nu}{}^{\rho\sigma}\Sigma^i_{\rho\sigma} = 2F^i_{\mu\nu}$$

Here $F^i = dA^i + \frac{1}{2}\epsilon^{ijk}A^j \wedge A^k$ is the curvature of the SO(3) connection A

We have shown that the curvature of A encodes the self-dual part of Riemann with respect to a pair of indices

<u>Corollary:</u> $R_{\mu\nu} = \Lambda g_{\mu\nu} \quad \Leftrightarrow \quad {}_{-}Riem_{+} = 0 \quad \Leftrightarrow \quad F^i = M^{ij}\Sigma^j$ for some (symmetric) 3x3 matrix M

Thus, Einstein condition becomes a very simple and natural condition on the curvature of A

# Field equations and the action

The full set of field equations in this formalism

$$\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$$
$$d\Sigma^i + \epsilon^{ijk} A^j \wedge \Sigma^k = 0$$
$$F^i = \left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right)\Sigma^j$$

All equations are written only using the exterior derivative

They can be compared with the usual frame formalism

$$de^I + \omega^I{}_J \wedge e^J = 0$$

Cartan's first structure equation

$$\epsilon^{IJKL} e^J \wedge F^{KL} = 0$$

Ricci-flatness

Come from Plebanski action

$$S[\Sigma, A, \Psi] = \int \Sigma^i F^i - \frac{1}{2}(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij})\Sigma^i \wedge \Sigma^j$$

Einstein-Cartan action (with zero Lambda)

$$S[e, \omega] = \int \epsilon^{IJKL} e^I \wedge e^J \wedge F^{KL}$$

Why are 2-forms any superior to the frame formalism?

The answer is in the types of operators that appear in linearised theory

Also produces much more useful perturbation theory

Also this formalism is ideally adapted to describe instantons

# YM self-duality equations

Linearisation of YM self-duality equations $F(A)_+ = 0$ gives rise to a famous complex of differential operators

$$\Lambda^0(\mathfrak{g}) \xrightarrow{d_A} \Lambda^1(\mathfrak{g}) \xrightarrow{d_A^+} \Lambda^2_+(\mathfrak{g})$$

This complex is elliptic - the complex of symbols is exact

Adding the adjoint (gauge-fixing) one gets an elliptic operator

$$\Lambda^1(\mathfrak{g}) \xrightarrow{d_A^+ + (d_A)^*} \Lambda^0(\mathfrak{g}) \oplus \Lambda^2_+(\mathfrak{g}).$$

This operator is in fact a twisted Dirac operator $\mathfrak{g} \otimes S_+ \otimes S_- \to \mathfrak{g} \otimes S_+ \otimes S_+$

and plays the key role in understanding the local properties of the instanton moduli space

There are analogs of all of this for the case of gravity!

But needs the formalism of 2-frames!

# Gravitational instantons

We define gravitational instantons as Riemannian metrics that for which one of the halves of the Riemann curvature vanishes

$$Riem_+ = 0$$

These are automatically Einstein and Ricci-flat, because Einstein condition is $\,_-Riem_+ = 0$

and the scalar curvature is a component of $\,_+Riem_+ = 0$

The 2-frame formalism is ideally suited for the description of gravitational instantons

Recall that the curvature F of the SO(3) connection A encodes precisely $Riem_+$

These means that instanton metrics are encoded by 2-frames $\Sigma$ whose canonical connection is flat

It is then always possible to choose the 2-frame so that $A = 0$ or $\boxed{d\Sigma^i = 0}$

This gives a simple first-order in derivatives description of gravitational instantons

This is well-known, for example Eguchi-Hanson metric was obtained precisely from the condition $A = 0$

# HyperKahler

The described gravitational instantons are actually hyperKahler metrics

Riemannian manifolds that possess 3 complex structures satisfying the algebra of imaginary quaternions

These 3 complex structures are just our $\Sigma^i$ with one of the indices raised $\Sigma^i_\mu{}^\nu$

Another way to think about these manifolds is as those with an SU(2) structure

The globally defined triple of 2-forms $\Sigma^i$ is stabilised by an SU(2) subgroup of SO(4)

For each of the complex structures, the 2-forms define the Kahler form as well as the top holomorphic form

$$\omega = \Sigma^3 \qquad \Omega = \Sigma^1 + i\,\Sigma^2$$

The subgroup of SO(4) that preserves the Kahler form is U(2)

The subgroup of SO(4) that in addition preserves the top holomorphic form is SU(2)

# Action principle for grav. instantons

There is a simple action principle whose critical points are gravitational instantons

Effectively, it arises by dropping a term in Plebanski action

2106.01397 [hep-th]

With Skvortsov

$$S_{SD}[\Sigma, A, \Psi] = \int \Sigma^i dA^i - \frac{1}{2}\Psi^{ij}\Sigma^i \wedge \Sigma^j$$

The variation with respect to A gives the equation that the 2-frame is closed. We can then parametrise (locally)

$\Sigma^i = \Sigma_0^i + da^i$ where $a^i \in E \otimes \Lambda^1$ and $\Sigma_0^i$ is some hyperKahler background

This gives

$$S[a, \Psi] = -\frac{1}{2}\int \Psi^{ij}(\Sigma_0^i + da^i) \wedge (\Sigma_0^j + da^j)$$

This is a good point for instanton sector perturbation theory

In particular, one can compute the same helicity Berends-Giele current using this perturbation theory

This current is the main building block of the gravity MHV amplitudes

# Plebanski complex

2502.19027 [math.DG]

Theorem: Linearisation of the instanton equations gives rise to the following complex of differential operators

This side is the diffeomorphism gauge $\longrightarrow$ $TM \xrightarrow{d_1} S \xrightarrow{d_2} E \otimes \Lambda^1 \xrightarrow{d_3} E$ $\longleftarrow$ This side is SO(3) gauge

Where

$$d_1 : TM \to S, \qquad d_1\xi = d(\xi \lrcorner \Sigma^i).$$

$$d_2 : S \to E \otimes \Lambda^1, \qquad d_2\sigma^i = \frac{1}{2} J_1^{-1}(\star d\sigma^i).$$

$$d_3 : E \otimes \Lambda^1 \to E, \qquad d_3 a^i = da^i\Big|_3 \equiv \epsilon^{ijk}\Sigma^j \wedge da^k / v_\Sigma.$$

And S is the tangent space to the space of 2-frames $\Sigma$

$$S = (E \otimes \Lambda^2)_{1+3+9}$$

This complex is elliptic, the sequence of dimensions is $4 \to 13 \to 12 \to 3$

Its appropriate gauge-fixing (adding adjoints) can be shown to be related to the direct sum of two Dirac operators

$$D_4 : S_+ \otimes S_+ \to S_- \otimes S_+, \qquad D_{12} : S_- \otimes S_- \otimes S_+^2 \to S_+ \otimes S_- \otimes S_+^2.$$

This complex should be as useful as the YM instanton one!

# Scattering amplitudes

Plebanski formalism gives rise to a very useful perturbative expansion

Can compute gravity MHV amplitudes using the technology of Berends-Giele currents

This is the same technique by which MHV gluon amplitudes - Parke-Taylor formula - have been derived

I will now sketch this calculation, starting with the YM case

The Berends-Giele currents are defined to be the sum of all Feyman diagrams with all legs on-shell, apart from one off-shell leg

The power of this approach to perturbative calculations is in the fact that previously computed BG currents

can be sewn to compute the BG currents for a larger number of particles

Particularly powerful is the application of this technique to BG currents with all particles of the same helicity

or to BG currents with all but one particles of the same helicity

In these cases the arising recursion relations are sufficiently simple to be solvable exactly

# Gluon MHV amplitudes

In the case of gluons, direct application of YM Feynman rules gives the following recursion for all-plus BG current

$$J(1\dots n) = \frac{1}{\Box}\left(\sum_{m=1}^{n-1} (q|1\dots m|m+1\dots n|q) J(1\dots m) J(m+1\dots n)\right).$$

This is the recursion for the scalar part of the currents

All currents are vector-valued, but the vector structure follows the same pattern, so only need the scalar part

The solution is

$$J(1\dots n) = \frac{1}{(q1)(12)\dots(n-1\ n)(nq)}.$$

The all-but-one-plus BG current satisfies a more complicated recursion

$$J(1|2\dots n) = \frac{1}{\Box}\left(\frac{(q|2\dots n|p]}{[1p]} J(2\dots n) + \sum_{m=2}^{n-1} (q|1\dots m|m+1\dots n|q) J(1|2\dots m) J(m+1\dots n)\right).$$

The solution is

$$J(1|2\dots n) = J(2\dots n)\left(\frac{[2p]}{[12][1p]} + \sum_{m=2}^{n-1} \frac{(q|1\dots m|1\dots m+1|q)}{(1\dots m)^2 (1\dots m+1)^2}\right).$$

Only the last term in this sum matters for MHV amplitudes, which are then extracted

BG paper of 1987

# Graviton MHV amplitudes

Gravity BG currents can be computed using the same philosophy, starting with the perturbative expansion of Plebanski action

One finds that only a single cubic vertex matters for these calculations $\int eeda$

The recursion relation for the scalar part of the all-plus BG currents

$$J(\mathcal{K}) = \frac{1}{\square}\left(\sum_{|\mathcal{I}|<|\mathcal{J}|,\mathcal{I}\cup\mathcal{J}=\mathcal{K}} (q|\mathcal{I}|\mathcal{J}|q)^2 J(\mathcal{I})J(\mathcal{J})\right).$$

The solution is a sum over graphs (spanning trees)

$$J(\mathcal{K}) = \sum_{\Gamma_{\mathcal{K}}} J(\Gamma_{\mathcal{K}}), \quad J(\Gamma_{\mathcal{K}}) = \prod_{i\in\mathcal{K}} (qi)^{2\alpha_i-4} \prod_{\langle jk\rangle\in\mathcal{K}} \frac{[jk]}{(jk)}.$$

The recursion relation for the scalar part of the all-but-one-plus BG currents

$$J(1|\mathcal{K}) = \frac{1}{\square}\left(\frac{(q|2\ldots n|p]^2}{[1p]^2} J(\mathcal{K}) + \sum_{|\mathcal{I}|<|\mathcal{J}|,\mathcal{I}\cup\mathcal{J}=\mathcal{K}} (q|\mathcal{I}|\mathcal{J}|q)^2 J(1|\mathcal{I})J(\mathcal{J})\right).$$

Only the last term in this sum matters for MHV amplitudes, which are then extracted

The solution is again a sum over graphs

$$J(1|\mathcal{K}) = \sum_{\Gamma_{\mathcal{K}}} J(1|\Gamma_{\mathcal{K}}), \qquad J(1|\Gamma_{\mathcal{K}})/J(\Gamma_{\mathcal{K}}) \equiv \Phi(\Gamma_{\mathcal{K}}) = \sum_{\Gamma_{\mathcal{I}}\subseteq\Gamma_{\mathcal{K}}, i=|\mathcal{I}|} \phi_i(\Gamma_{\mathcal{I}}).$$

Complete parallel with the YM case

# Conclusions

There is a very powerful formalism that describes 4D Riemannian geometry in terms of triples of 2-forms

These 2-forms are not (normally) globally defined

One can instead think about a single 2-frame field $\Sigma : E \to \Lambda^+ \subset \Lambda^2$

The field equations in this formalism are in terms of the exterior derivative operator

Incredibly powerful formalism from which the description of the instanton sector of GR is directly obtainable

Many non-trivial facts about 4D Einstein manifolds are obtained in this formalism in a much simpler way

The linearisation of field equations gives rise to an elliptic complex - analogue of the YM instanton complex

Gravity MHV amplitudes can be computed analogously to the YM case

The next step for this formalism is to tackle a problem that is too difficult in normal metric GR

- Numerical Relativity?
- Gravitational self-force?

# Thank you!