



Worldline HSG

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Worldline Higher Spin Gravity

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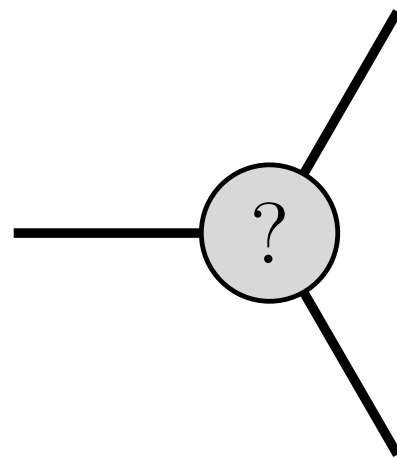
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Microscopic description of Higher Spin Gravity (HSG)?

- HSG [Vasiliev -90-] as Toy Model of Tensionless String
 - **Similarity**: ∞ many massless fields
 - **Difference**: only symmetric tensor fields in HSG
- A microscopic (worldsheet) description of HSG?
 - Closed or open string?
 - Brane partons [Enquist-Sundell 06]

Worldline (WL)

- WL action for any particle species (constrained system) [e.g. Gershun+ 79, Howe+ 88, Kuzenko+ 95&21, Lyakhovich+ 99, Bastianelli+ 08-, Andrzejewski+ 08, Basile+ 24]
- Describe only free theory (on a background)
- For interactions, add junctions and assign **junction conditions**



- Both vectorial and twistorial descriptions available

WL Twistor

- Particular dimensions; Simpler (fewer constraints)
- Connection to spacetime through **incidence relation**
- WL twistor for massless field in AdS [Cederwall 00, Arvanitakis+ 17-; Claus+ 99, Fedoruk+ 14, Mezincescu+ 15]
- HSG spectrum [Cederwall 00]
 - Remove the **spin constraint**
 - Unconstrained action for AdS₄

WL Twistor in AdS₄

$$S = \int \Omega_{AB} \mathbf{Z}^A \cdot d\mathbf{Z}^B$$

- WL variables

$$\mathbf{Z}^A = \begin{pmatrix} \xi^\alpha \\ \pi_\alpha \end{pmatrix} = \begin{pmatrix} \xi_1^\alpha & \xi_2^\alpha \\ \pi_{1\alpha} & \pi_{2\alpha} \end{pmatrix}, \quad \text{with} \quad \alpha = 1, 2$$

- Symplectic form

$$\Omega_{AB} = \frac{1}{2} \begin{pmatrix} 0 & -\delta_\beta^\alpha \\ \delta_\alpha^\beta & 0 \end{pmatrix}$$

- After integration by part

$$S = - \int d\pi_\alpha \cdot \xi^\alpha$$

WL Twistor in AdS₄

$$S = \int \Omega_{AB} \mathbf{Z}^A \cdot d\mathbf{Z}^B$$

- WL variables

$$\mathbf{Z}^A = \begin{pmatrix} \xi^\alpha \\ \pi_\alpha \end{pmatrix} = \begin{pmatrix} \xi_1^\alpha & \xi_2^\alpha \\ \tau_{1\alpha} & \tau_{2\alpha} \end{pmatrix}, \quad \text{with } \alpha = 1, 2$$

Internal or dual indices

- Symplectic form

$$\Omega_{AB} = \frac{1}{2} \begin{pmatrix} 0 & -\delta_\beta^\alpha \\ \delta_\alpha^\beta & 0 \end{pmatrix}$$

- After integration by part

$$S = - \int d\pi_\alpha \cdot \xi^\alpha$$

Quantized WL

$$\langle \Psi_2 | \xrightarrow[eI]{sI, I} | \Psi_1 \rangle$$

- Path Integral

$$\langle \Psi_2 | \Psi_1 \rangle = \int D\mathbf{Z} \langle \Psi_2 | \boldsymbol{\pi}(eI) \rangle \langle \boldsymbol{\pi}(sI) | \Psi_1 \rangle \exp \left(-i \int_I d\boldsymbol{\pi}_\alpha \cdot \boldsymbol{\xi}^\alpha \right)$$

- Due to trivial Hamiltonian, the above reduces to

$$\langle \Psi_2 | \Psi_1 \rangle = \int \frac{d^4 \boldsymbol{\pi}(\tau_0)}{(2\pi)^2} \langle \Psi_2 | \boldsymbol{\pi}(\tau_0) \rangle \langle \boldsymbol{\pi}(\tau_0) | \Psi_1 \rangle$$

- Simpler in canonical quantization, but useful to keep the path integral picture in mind

Quantized WL

- 1-ptcle Hilbert Space: Space of functions of π
- (Removed) Spin constraint

$$\mathcal{S} = \frac{1}{2} \left(\pi_{1\alpha} \frac{\partial}{\partial \pi_{2\alpha}} - \pi_{2\alpha} \frac{\partial}{\partial \pi_{1\alpha}} \right)$$

! Eigenvalue $\pm s$ correspond to the same spin s irrep

- AdS₄ isometry (3d conformal symmetry)

$$P_{\alpha\beta} = \pi_\alpha \cdot \pi_\beta, \quad K^{\alpha\beta} = -\frac{\partial}{\partial \pi_\alpha} \cdot \frac{\partial}{\partial \pi_\beta}, \quad L_\alpha{}^\beta = -i \left(\pi_\alpha \cdot \frac{\partial}{\partial \pi_\beta} + \delta_\alpha^\beta \right)$$

Free WL

Construct field operator
from covariance
(à la Weinberg)

- Free Scalar
- Free Spin s



AdS₄ Covariance

- Use translational covariance

$$\mathcal{O}(x, \rho) = e^{\frac{i}{2} x^{\alpha\beta} P_{\alpha\beta}} e^{-i \rho D} \mathcal{O}(o) e^{i \rho D} e^{-\frac{i}{2} x^{\alpha\beta} P_{\alpha\beta}}$$

- Poincaré chart $ds^2 = d\rho^2 + e^{2\rho} dx^2 = \frac{dz^2 + dx^2}{z^2}$
- Use Lorentz covariance of $\mathcal{O}(o)$
- Free theory: $\mathcal{O}(o) |\Omega\rangle$ belongs to the 1-ptcle Hilbert space

Free Scalar

- AdS₄ covariance

- Translation $\langle \pi | \Phi(x, z) | \Omega \rangle = e^{\frac{i}{2} x^{\alpha\beta} \pi_{\alpha} \cdot \pi_{\beta}} z \langle \sqrt{z} \pi | \Phi(o) | \Omega \rangle$

- Lorentz $0 = \langle \pi | [J_{\alpha\beta}, \Phi(o)] | \Omega \rangle = i \pi_{(\alpha} \cdot \frac{\partial}{\partial \pi^{\beta)}} \langle \pi | \Phi(o) | \Omega \rangle$

- Field operator $\langle \pi | \Phi(x, z) | \Omega \rangle = e^{\frac{i}{2} x^{\alpha\beta} \pi_{\alpha} \cdot \pi_{\beta}} z \left[c_+ e^{+i z \langle \pi_1 \pi_2 \rangle} + c_- e^{-i z \langle \pi_1 \pi_2 \rangle} \right]$

- Undetermined constants c+ and c-

- Gaussian function in π

- KG eq $[(z \partial_z)^2 - 3 z \partial_z + z^2 \partial_x^2 - m^2] \langle \pi | \Phi(x, z) | \Omega \rangle = 0$ with $m^2 = -2$

Free Scalar

- 2pt function
- Gaussian integral

$$\langle \Omega | \Phi(x, z) \Phi(x', z') | \Omega \rangle = \int \frac{d^4 \pi}{(2\pi)^2} \langle \Omega | \Phi(x, z) | \pi \rangle \langle \pi | \Phi(x', z') | \Omega \rangle = \frac{\frac{|c_+|^2 + |c_-|^2}{2}}{\text{Dist}^2(x, z | x', z')} + \frac{\frac{c_+^* c_- + c_-^* c_+}{2}}{\text{Dist}^2(x, -z | x', z')}$$

- $\text{Dist}^2(x, z | x', z') = \frac{(x - x')^2 + (z - z')^2}{2 z z'}$
- Dirichlet BC $c_+ = c_-$, Neumann BC $c_+ = -c_-$
- Mirror or Anti-podal contribution

$$(x, z)_{\text{Mirror}} = (x, -z) = (x, z)_{\text{Antipodal}}$$

Free Spin s

- Spin s Weyl tensor (gauge invariant)
 - Traceless s -derivatives of the Fronsdal field
 - Split into self-dual and anti-self-dual parts $C^{(+s)}$ and $C^{(-s)}$
- Transverse components
 - Rank- $2s$ $sl(2, \mathbb{R})$ tensors $C_{\alpha_1 \dots \alpha_{2s}}^{(+s)}$ and $C_{\alpha_1 \dots \alpha_{2s}}^{(-s)}$
 - Polarization spinor

$$C^{(+)}(x, z; v) = \sum_{2s=0}^{\infty} \frac{1}{(2s)!} v^{\alpha_1} \dots v^{\alpha_{2s}} C_{\alpha_1 \dots \alpha_{2s}}^{(+s)}(x, z)$$

Free Spin s

- AdS4 Covariance

$$\langle \boldsymbol{\pi} | C^{(+)}(x, z; v) | \Omega \rangle = e^{\frac{i}{2} x^{\alpha\beta} \boldsymbol{\pi}_\alpha \cdot \boldsymbol{\pi}_\beta} z \left[c_+^{(+)} e^{+i z \langle \pi_1 \pi_2 \rangle + \sqrt{z} \langle (\pi_1 + i \pi_2) v \rangle} + c_-^{(+)} e^{-i z \langle \pi_1 \pi_2 \rangle + \sqrt{z} \langle (\pi_1 - i \pi_2) v \rangle} \right]$$

$$\langle \boldsymbol{\pi} | C^{(-)}(x, z; v) | \Omega \rangle = e^{\frac{i}{2} x^{\alpha\beta} \boldsymbol{\pi}_\alpha \cdot \boldsymbol{\pi}_\beta} z \left[c_+^{(-)} e^{+i z \langle \pi_1 \pi_2 \rangle + \sqrt{z} \langle (\pi_1 - i \pi_2) v \rangle} + c_-^{(-)} e^{-i z \langle \pi_1 \pi_2 \rangle + \sqrt{z} \langle (\pi_1 + i \pi_2) v \rangle} \right]$$

- Satisfy Bargmann eq

- Gaussian function

- Dirichlet BC $c_+^{(+)} = c_-^{(-)} = c_+$, $c_-^{(+)} = c_+^{(-)} = c_-$

- Helicity $\neq$ Spin constraint eigenvalue

Free Spin s

- 2pt function

$$\langle \Omega | C^{(-)}(x, z; v) C^{(+)}(x', z'; v') | \Omega \rangle = \frac{(|c_+|^2 + |c_-|^2) e^{-2i \sqrt{zz'}} \langle v | P(x, z; x', z') | v' \rangle}{2 \text{Dist}^2(x, z; x', z')}$$

$$\langle \Omega | C^{(+)}(x, z; v) C^{(+)}(x', z'; v') | \Omega \rangle = \frac{(|c_+|^2 + |c_-|^2) e^{-2i \sqrt{zz'}} \langle v | \bar{P}(x, -z; x', z') | v' \rangle}{2 \text{Dist}^2(x, -z; x', z')}$$

- $P_{\alpha\beta}(x, z; x', z') = \frac{(x - x')_{\alpha\beta} + i(z - z') \epsilon_{\alpha\beta}}{(x - x')^2 + (z - z')^2}$

- Constants c+ and c- are not fully fixed
- Helicity flips after the reflection off the boundary

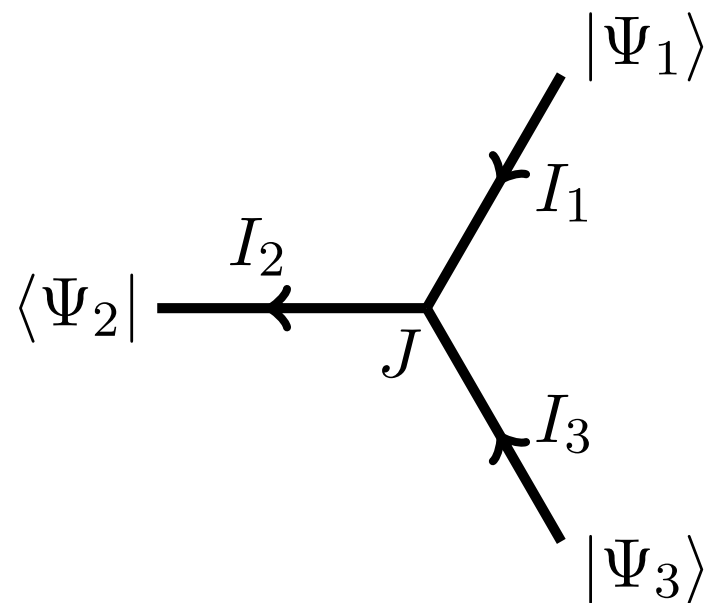
Interacting WL

- Junction Condition
- Vertex Operator



Junction Condition

$$A = \int D\mathbf{Z} \langle \Psi_2 | \pi(eI_2) \rangle \langle \pi(sI_1) | \Psi_1 \rangle \langle \pi(sI_3) | \Psi_3 \rangle \exp \left(i \int_{I_1 \cup I_2 \cup I_3} \Omega_{AB} \mathbf{Z}^A \cdot d\mathbf{Z}^B \right)$$



- Half (polarization) of $\mathbf{Z}^A(eI_1)$, $\mathbf{Z}^A(sI_2)$, and $\mathbf{Z}^A(eI_3)$ should be related at

$$J = eI_1 = sI_2 = eI_3$$

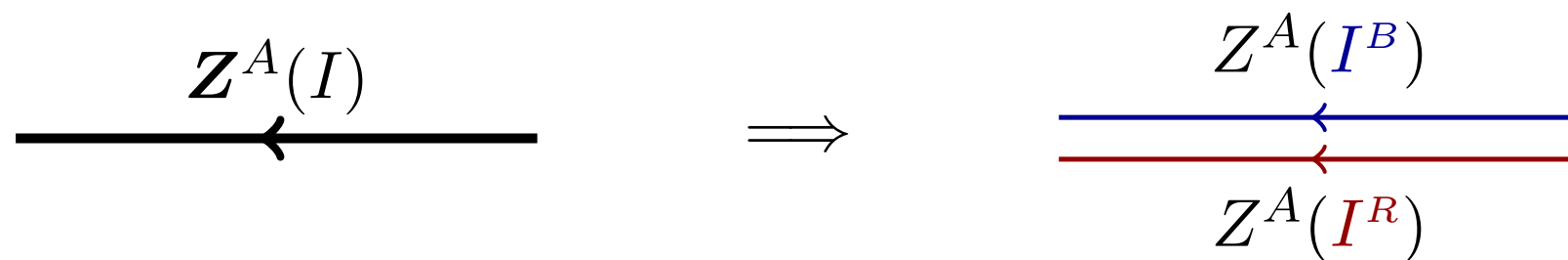
Junction Condition

- WL theory is free
 - No built-in information of junction condition in general
- In fact, the situation is the same in String Theory
 - WS theory is free
 - Junction condition is **geometrically** prescribed
- Maybe, an analogous junction condition exists for WL twistor

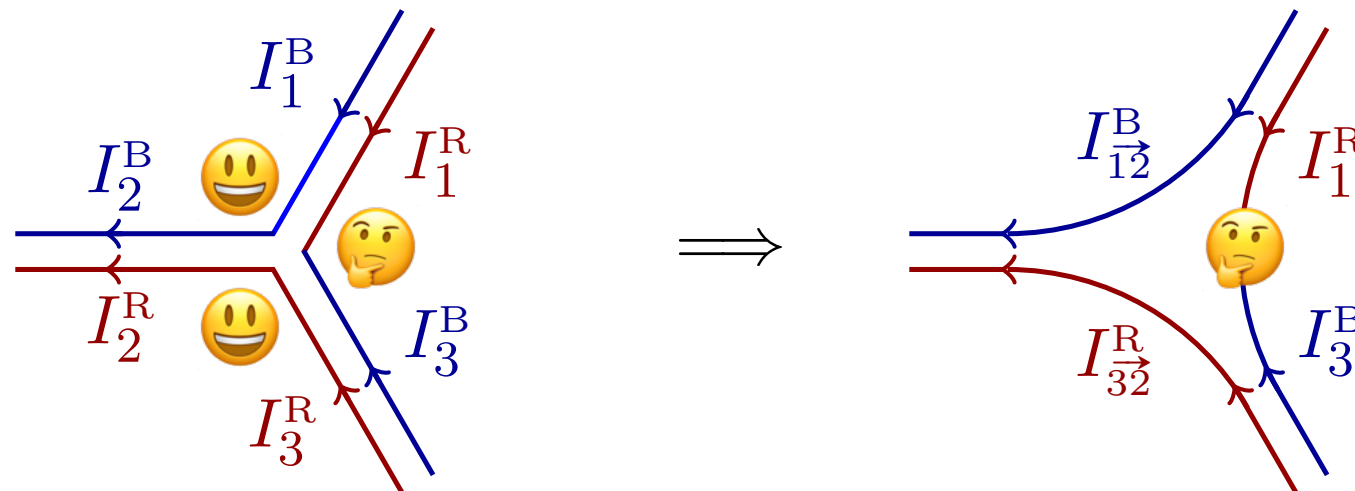
Junction Condition

- Double line interpretation

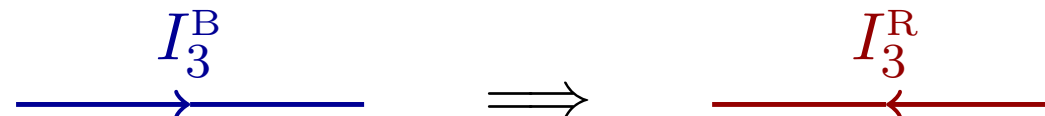
$$\begin{aligned} \int D\mathbf{Z} \, e^{i \int_{I_i} \Omega_{AB} \mathbf{Z}^A \cdot d\mathbf{Z}^B} &= \int DZ_1 DZ_2 \, e^{i \int_{I_i} \Omega_{AB} Z_1^A dZ_1^B + i \int_{I_i} \Omega_{AB} Z_2^A dZ_2^B} \\ &= \int DZ \, e^{i \int_{I_i^B} \Omega_{AB} Z^A dZ^B + i \int_{I_i^R} \Omega_{AB} Z^A dZ^B}, \end{aligned}$$



Junction Condition



- Orientation Flip



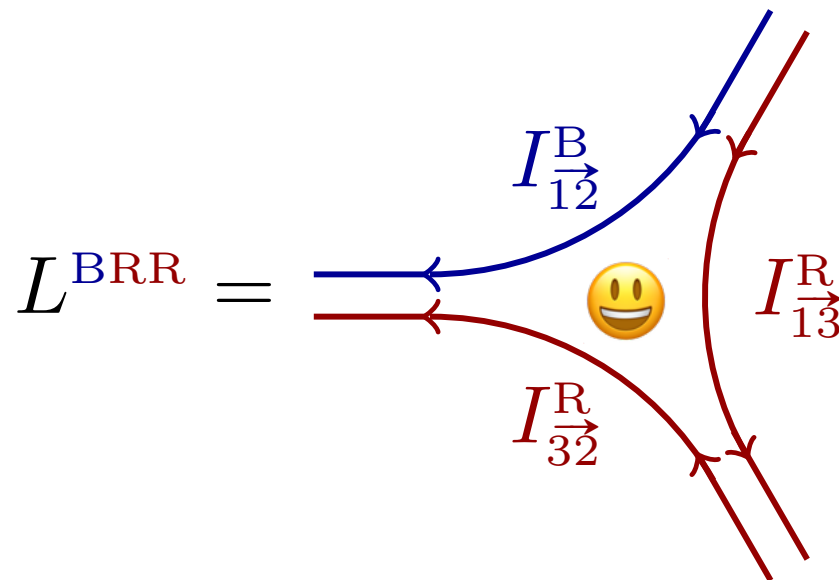
$$\int_{I_3^B} \Omega_{AB} Z^A dZ^B = - \int_{I_3^R} \Omega_{AB} Z^A dZ^B$$

- Fix the sign with $Z^A \rightarrow i Z^A$

Junction Condition

$$A = \int D\xi^\alpha D\pi_\alpha \langle \Psi_2 | \pi(e\vec{I}_{12}^B), \pi(e\vec{I}_{32}^R) \rangle \langle \pi(s\vec{I}_{12}^B), \pi(s\vec{I}_{13}^R) | \Psi_1 \rangle \times$$

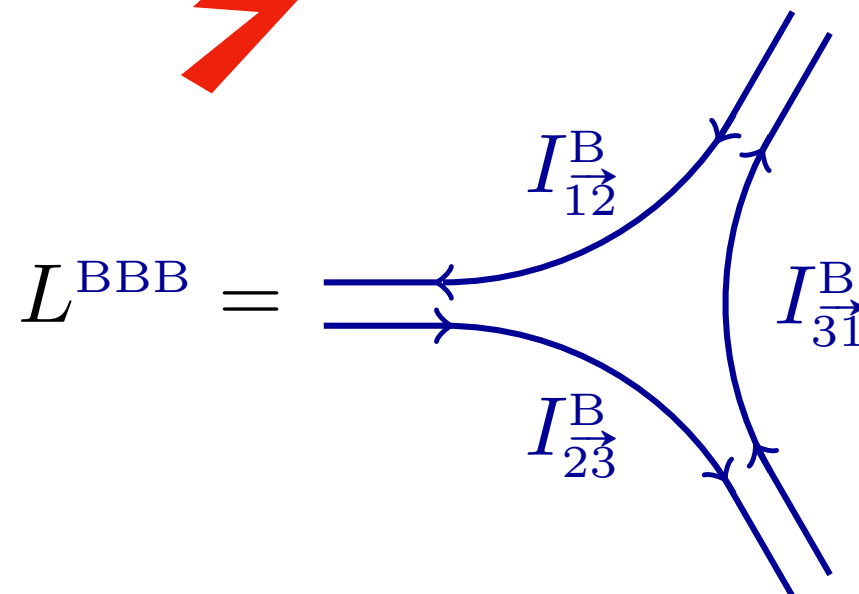
$$\times \langle i\pi(e\vec{I}_{13}^R), \pi(s\vec{I}_{32}^R) | \Psi_3 \rangle \exp\left(-i \int_{L^{\text{BRR}}} d\pi_\alpha \xi^\alpha\right)$$



Junction Condition

- Equivalently

$$A = \int D\xi^\alpha D\pi_\alpha \langle \Psi_2 | \pi(e\vec{I}_{12}^B), \circledast \pi(s\vec{I}_{23}^B) \rangle \langle \pi(s\vec{I}_{12}^B), \circledast \pi(e\vec{I}_{31}^B) | \Psi_1 \rangle \times \\ \times \langle \pi(s\vec{I}_{31}^B), \circledast \pi(e\vec{I}_{23}^B) | \Psi_3 \rangle \exp \left(-i \int_{L^{BBB}} d\pi_\alpha \xi^\alpha \right)$$



Vertex Operator

- For manifest crossing symmetry, we introduce

$$\langle \pi_1 | \hat{V}_\Psi | \pi_2 \rangle \equiv \langle \pi_1, \dot{i} \pi_2 | \Psi \rangle \quad \text{state/operator correspondence}$$

- Twisted Adjoint to Adjoint Representation

$$X \triangleright |\Psi\rangle = X |\Psi\rangle, \quad X \triangleright \hat{V}_\Psi = [X, \hat{V}_\Psi]$$

- Trace

$\text{Tr} \left(\hat{V}_{\Psi_2} \hat{V}_{\Psi_1} \right)$
“Feynman”

$\langle \Psi_2 | \Psi_1 \rangle = -\text{Tr} \left(\hat{\tau} \hat{V}_{\Psi_2}^\dagger \hat{V}_{\Psi_1} \right)$
“Wightman”

$\hat{\tau} |\pi\rangle = |-\pi\rangle$

Vertex Operator

- Matrix element

$$\begin{aligned}\langle \pi_1 | \hat{V}(x, z) | \pi_2 \rangle &\equiv \langle \pi_1, i\pi_2 | \Phi(x, z) | \Omega \rangle \\ &= e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta})} z \left[c_+ e^{-z \langle \pi_1 \pi_2 \rangle} + c_- e^{+z \langle \pi_1 \pi_2 \rangle} \right]\end{aligned}$$

- Operator form (using Wigner-Weyl transform)

$$V(x, z | \xi, \pi) = c_+ z \delta^2(\xi^\alpha - x^{\alpha\beta} \pi_\beta - iz \pi^\alpha) + c_- z \delta^2(\xi^\alpha - x^{\alpha\beta} \pi_\beta + iz \pi^\alpha)$$

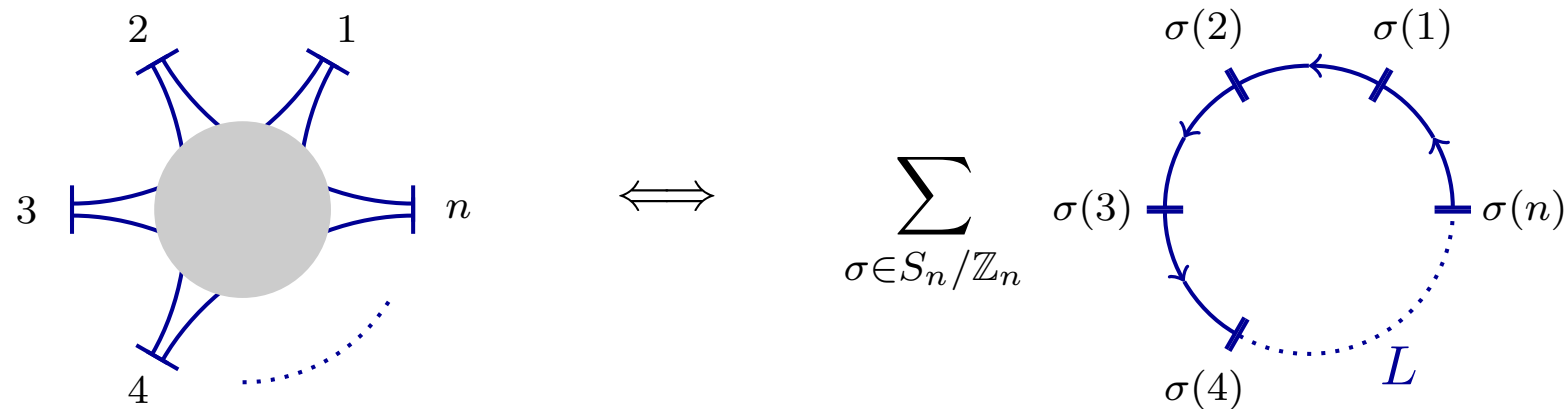
- Incidence relation?

WL HSG



WL HSG

$$\begin{aligned} & \langle C(x_1, z_1; v_1) \cdots C(x_n, z_n; v_n) \rangle_{\text{connected}}^{\text{HSG}} \\ &= \sum_{\sigma \in S_n / \mathbb{Z}_n} \frac{1}{g_{\text{HSG}}^2} \langle \hat{V}(x_{\sigma(1)}, z_{\sigma(1)}; v_{\sigma(1)}) \cdots \hat{V}(x_{\sigma(n)}, z_{\sigma(n)}; v_{\sigma(n)}) \rangle_{\text{WL}} \end{aligned}$$



- WL correlation fn as Trace

$$\begin{aligned} \langle \hat{V}(x_1, z_1; v_1) \cdots \hat{V}(x_n, z_n; v_n) \rangle_{\text{WL}} &= \int D\xi^\alpha D\pi_\alpha V(x_1, z_1; v_1) \cdots V(x_n, z_n; v_n) e^{-i \int_L d\pi_\alpha \xi^\alpha} \\ &= \text{Tr} \left(\hat{V}(x_1, z_1; v_1) \cdots \hat{V}(x_n, z_n; v_n) \right) \end{aligned}$$

WL HSG

- Vertex operator fixed by dual parity map

$$\text{Type A:} \quad \langle \pi_1 | \hat{V} | -\pi_2 \rangle = + \langle \pi_1 | \hat{V} | \pi_2 \rangle$$

$$\text{Type B:} \quad \langle \pi_1 | \hat{V} | -\pi_2 \rangle = - \langle \pi_1 | \hat{V} | \pi_2 \rangle$$

- Type A

$$\begin{aligned} & \langle \pi_1 | \hat{V}_A(x, z; v) | \pi_2 \rangle \\ &= g_{\text{HSG}} z e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta})} \cosh(z \langle \pi_1 \pi_2 \rangle) \cosh(\sqrt{z} \langle \pi_1 v \rangle) \cosh(\sqrt{z} \langle \pi_2 v \rangle) \end{aligned}$$

- Type B

$$\begin{aligned} & \langle \pi_1 | \hat{V}_B(x, z; v) | \pi_2 \rangle \\ &= i g_{\text{HSG}} z e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta})} \cosh(z \langle \pi_1 \pi_2 \rangle) \sinh(\sqrt{z} \langle \pi_1 v \rangle) \sinh(\sqrt{z} \langle \pi_2 v \rangle) \end{aligned}$$

$$\langle \pi_1 | \hat{V}_B(x, z) | \pi_2 \rangle = i g_{\text{HSG}} z e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta})} \sinh(z \langle \pi_1 \pi_2 \rangle)$$

WL HSG

- AdS₄ Correlation functions

- Combination of Trace (Gaussian integral)

- Introduce $\langle \pi_1 | \hat{G}(x, z; v^L, v^R) | \pi_2 \rangle = e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta}) + z \langle \pi_1 \pi_2 \rangle + \langle \pi_1 v^L \rangle + \langle \pi_2 v^R \rangle}$

- Trace $\text{Tr} \left[\hat{G}(x_1, z_1; v_1^L, v_1^R) \cdots \hat{G}(x_n, z_n; v_n^L, v_n^R) \right] = \frac{\exp \left(\frac{\mathcal{L} + i \mathcal{P}}{2} \right)}{\sqrt{x_{12}^2 x_{23}^2 \cdots x_{n1}^2}}$

- Expansion in z
$$\mathcal{L}^{(m)} = (-i)^m \sum_{\substack{i_1, i_2, \dots, i_m \\ \text{with } i_{p+1} = i_p \pm 1, i_m = i_0}} z_{i_1} \cdots z_{i_m} \frac{1}{m} \text{tr} \left(x_{i_1 i_1+1}^{-1} x_{i_2 i_2+1}^{-1} \cdots x_{i_m i_m+1}^{-1} \right)$$

$$\mathcal{P}_{i_0 i_m}^{(m)} = (-i)^m \sum_{\substack{i_0, i_1, \dots, i_m \\ \text{with } i_{p+1} = i_p \pm 1}} z_{i_1} \cdots z_{i_m} \langle v_{i_0}^L + v_{i_0+1}^R | x_{i_0 i_0+1}^{-1} x_{i_1 i_1+1}^{-1} \cdots x_{i_m i_m+1}^{-1} | v_{i_m}^L + v_{i_m+1}^R \rangle$$

- Are these all relevant?

WL HSG

- Similarities with String Theory
 - Vertex operators satisfy free equation
 - Fields become free only at asymptotic infinity

String theory	HSG worldline theory
Worldsheet parameters τ, σ Position variable $X^\mu(\tau, \sigma)$ String state $\Psi[X^\mu(\tau_0, \sigma)]$ at fixed τ_0 Momentum p_μ Plane wave $e^{i p_\mu X^\mu(\tau, \sigma)}$	Worldline parameter τ Twistor variable $\pi_\alpha(\tau)$ HSG state $\Psi(\pi_\alpha(\tau_0))$ at fixed τ_0 AdS ₄ point $(x^{\alpha\beta}, z)$ Delta function $\delta^2(\xi^\alpha(\tau) - x^{\alpha\beta} \pi_\beta(\tau) \pm i z \pi^\alpha(\tau))$

WL HSG

- Boundary limit of Type A/B WL HSG correlation functions reproduce free boson/fermion CFT correlation functions
- ex. type A:

$$\begin{aligned} & \text{Tr} \left[\frac{1}{z_1} \hat{V}_A \left(x_1, z_1; \frac{v_1}{\sqrt{z_1}} \right) \cdots \frac{1}{z_n} \hat{V}_A \left(x_n, z_n; \frac{v_n}{\sqrt{z_n}} \right) \right] \\ &= g_{\text{HSG}}^n \frac{\prod_{i=1}^n \exp \left(\frac{i}{2} Q_i \right) \prod_{i=1}^n \cos(P_{i i+1})}{\sqrt{x_{12}^2 \cdots x_{n1}^2}} + \mathcal{O}(z_i) \end{aligned}$$

$$Q_i = \langle v_i | (x_{i i+1}^{-1} - x_{i i-1}^{-1}) | v_i \rangle, \quad P_{i i+1} = R_{i i+1}, \quad R_{ij} = \langle v_i | x_{i i+1}^{-1} \cdots x_{j-1 j}^{-1} | v_j \rangle$$

Remarks



CFT

- Higher Spin/O(N) CFT duality
 - Conjecture [Sezgin-Sundell 02, Klebanov-Polyakov 02]
 - Test [Giombi-Yin 10-, Colombo-Sundell 12, Didenko-Skvortsov 13, ...]
- CFT operators [Sezgin+ 17]

$$j_A(\mathbf{x}; v) = \cos(2e^{i\pi/4}\sqrt{U_1}) \cos(2e^{i\pi/4}\sqrt{-U_2}) \bar{\phi}_a(\mathbf{x}_1) \phi^a(\mathbf{x}_2) \Big|_{\mathbf{x}_1=\mathbf{x}_2=\mathbf{x}}$$

$$U_1 = \frac{1}{2} v^\alpha v^\beta \partial_{\alpha\beta}^1, \quad U_2 = \frac{1}{2} v^\alpha v^\beta \partial_{\alpha\beta}^2$$

$$\langle \pi_1 | \hat{V}_A(x, z; v) | \pi_2 \rangle = g_{\text{HSG}} z e^{\frac{i}{2} x^{\alpha\beta} (\pi_{1\alpha} \pi_{1\beta} - \pi_{2\alpha} \pi_{2\beta})} \cosh(z \langle \pi_1 \pi_2 \rangle) \cosh(\sqrt{z} \langle \pi_1 v \rangle) \cosh(\sqrt{z} \langle \pi_2 v \rangle)$$

Twisted Adjoint Eq

- Twisted adjoint field [Colombo-Sundell 12, Didenko-Skvortsov 13]

$$B(x, z; \mathbf{x}_i, \zeta_i, \bar{\zeta}_i | y, \bar{y}) = K(x, z, \mathbf{x}_i) e^{-i y_\alpha F^{\alpha\dot{\alpha}}(x, z, \mathbf{x}_i) \bar{y}_{\dot{\alpha}}} \left(e^{i(\zeta_i^\alpha y_\alpha + \theta)} + \text{c.c} + \left(\frac{\zeta \rightarrow -\zeta}{\bar{\zeta} \rightarrow -\bar{\zeta}} \right) \right)$$

$$K(x, z, \mathbf{x}_i) = \frac{z}{(x - \mathbf{x}_i)^2 + z^2}, \quad \zeta_i^\alpha = K(x, z, \mathbf{x}_i) \left(\frac{1}{\sqrt{z}} (x - \mathbf{x}_i)^{\alpha\beta} - \sqrt{z} i \epsilon^{\alpha\beta} \right) v_{i\beta}$$

$$F^{\alpha\dot{\alpha}}(x, z, \mathbf{x}_i) = - \left(\frac{2z}{(x - \mathbf{x}_i)^2 + z^2} (x - \mathbf{x}_i)^{\alpha\dot{\alpha}} + \frac{(x - \mathbf{x}_i)^2 - z^2}{(x - \mathbf{x}_i)^2 + z^2} i \epsilon^{\alpha\dot{\alpha}} \right)$$

- Bulk-to-Boundary propagator
- Trace

$$\text{Tr} \left(B(X; \mathbf{x}_1, Z_1 | Y) \star \kappa \star B(X, \mathbf{x}_2, Z_2 | Y) \star \cdots \star B(X; \mathbf{x}_n, Z_n | Y) \star \kappa \right)$$

- Bulk coord: x decouples and z set to a reference point

Brane Partons

[Enquist-Sundell 06]

- Emergent in tensionless limit of ST
- N singletons (2 singletons for HSG)
- Vertex operators in adjoint representation
- Vector variables with $Sp(2, \mathbb{R})$ constraints
- Topological String (Poisson sigma model)

Poisson Sigma Model

- WL model as Edge mode of PSM

$$S_{\text{PSM}} = \int_{\Sigma} \eta_A \wedge dZ^A + \frac{1}{2} \mu^{AB}(Z) \eta_A \wedge \eta_B$$

- Summation over WS topologies
- Fractional Branes?
 - Target space is $\mathbf{Z}_2$ orbifold (dual parity map)
 - Geometric counterpart of “multiparticle algebra” [Vasiliev 13&18]

Other Aspects

- Loops
 - as WL HSG, no loop unless we extend the prescription
 - PSM contains loops (ex. 1-loop = Annulus or Mobius)
- Unoriented theory
- Other future directions
 - dS, Flat (Chiral)
 - Deformation for theta dependence (CS matter CFT)?



Thank you